43. Search strategies as indicators of functional thinking

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1. Functional thinking

The concept of function is a "leading concept" in mathematics teaching. It is taught in secondary schools in a spiral approach in which students successively reach different levels of understanding (VOLLRATH 1984). These levels are characterized by abilities, for example that the student is able to realize in a situation that a quantity y depends upon another quantity x; or more specifically, the student knows that from an enlargement of x an enlargement of y will result. Such knowledge leads to assumptions when a new situation is investigated by the student. Once an observation differs from what is expected, learning can take place: a new assumption is made in order to explain the unexpected result.

To learn about functions and to be successful in using functions to solve problems requires a mental ability which can be characterized as follows:

(i) Dependences between variables can be stated, postulated, produced, and reproduced.

(ii) Assumptions about the dependence can be made, can be tested, and if necessary can be revised.

The mental activities described in (i) are fundamental for working on functions (FREUDENTHAL 1983). The activities in (ii) are typical for "mathematical thinking" (BURTON 1984). This ability can be called *functional thinking* which has been a key concept in mathematics education since the Meran Conference in 1905 (GUTZMER 1908). Many suggestions have been made by mathematics educators in Europe to promote functional thinking and there is some knowledge about the efficiency of these methods (e.g. ANDELFINGER 1981; HART 1981)

Knowledge about the *development* of functional thinking can be gained by psychological studies. From PIAGET's investigations it is known that the ability to discover the proportionality of a function develops in children (1958, 1968). He presented, for example, physical experiments and asked for the exact solutions of missing value problems which had to be found by calculation, and identified stages leading to "proportional reasoning". (Several studies attempted to fill gaps in the Piagetian model e.g. CASE 1979; RICCO 1982; SIEGLER 1976. For research on proportional reasoning we refer the reader to TOURNIAIRE and PULOS 1985). These stages can be interpreted as stages of functional thinking. They can be defined by abilities and by limitations, such as:

> The child knows that from an enlargement of x an enlargement of y will result. But the child is not able to discover that a doubling of x leads to a doubling of y.

These abilities and deficits become apparent in problem situations in which the student is asked to predict or to precalculate a result.

Refering to FREUDENTHAL (1983) proportional reasoning has a number aspect (ratio) and a function aspect (proportionality). The function aspect underlies the "building-up" strategies (HART 1981) which are frequently observed during childhood and adolescence (HART 1981; RICCO 1982). Changing strategies in an experiment can be understood as a result of changing assumptions about the dependence between the relevant quantities. When a missing value problem is presented in a physical experiment with an underlying quadratic function, many children assume proportionality, for example, and have difficulty overcoming this assumption (SUAREZ 1977). To be successful in such problems it is therefore important to discover properties "beyond proportionality".

The influence of teaching methods on the ability to form assumptions about functions, used in physics instruction was investigated by HÄUSSLER (1981). Experienced students typically start with the assumption of proportionality, but they also have a repertoire of further properties of functions which they can use for testing assumptions. Research on functional thinking should be aimed at yielding more information about the development of such a repertoire. Inexperienced children tend to use "fallback" strategies when confronted with a new situation (KARPLUS *et al.* 1983). There seems to be a hierarchy between the properties in the repertoire, which depends on the problems presented.

Most of the problems in these investigations had to be solved exactly by calculations. But there are many problems in mathematics instruction where solutions have to be found by approximation; e.g. calculating the zeros of $y = x^2 - 2$ by iteration, or approximating the ratio of the circumference to the diameter of a circle. There is a growing interest in such algorithms because of the increasing importance of the computer. They can be understood as goal-seeking processes, and the methods used are called *search strategies.*

Convergent search strategies very often result from properties of an underlying function. In these cases search strategies can be based on assumptions about the function. The main property used in search strategies for this type of problems is the *monotonic property.* Refering to FREUDENTHAL (1983) "aiming and hitting" is a fundamental source of the concept of function, which is closely connected with monotonicity.

Using the monotonic property in a PIAGETian experiment is an inappropriate strategy which has to be overcome. Therefore this property has not been found to be of much interest in the research on proportional reasoning. But for an "aiming and hitting" problem the monotonic property can be the basis for a successful strategy. An experiment with a goal-seeking problem can therefore provide knowledge about the discovery of the monotonic property as being a useful assumption for problem solving. Obviously a study of children's behaviour in such a problem situation can be a contribution to the knowledge about the development of functional thinking.

Two questions arise from these considerations:

(i) What information about functional thinking can be gained from search strategies?

(ii) Is it possible to identify stages of development leading towards the recognition and use of the monotonic property?

We tried to answer these questions in an investigation carried out from 1982 to 1984. K. KRAUSE, a student in mathematics education at Würzburg University, assisted me as experimenter.

2. Procedure

We wanted to work with children and young people. Therefore we decided on an experiment in which they could learn by doing. On the other hand we wanted to avoid an experiment which is used in physics instruction. So we built a wooden track (Figure 1), on which a steel ball could roll down in a channel and would be brought to rest by friction.

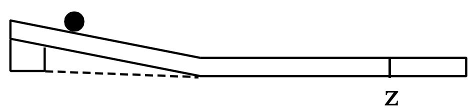

Fig. 1.

The subjects were 60 children aged from 4 to 15 years drawn from kindergarten, from youth-groups, and children using our play grounds, in the Würzburg area.

The children were told that they were to test a new toy. Their task was to find the point on the track at which the steel ball had to be placed so that it would roll as far as the marked point (Z) after being released. The children were told that they were allowed 20 trials, but that they had to find the correct starting point in as few trials as possible.

The result of each trial was marked on a strip, which was glued to the side of the track and could not be seen by the child. One such strip was used for each subject. The children could mark their starting points with a pencil if they wished.

Before the beginning of the experiment we checked that the ball always stopped at the same place when it was started at the same point several times. The gradient of the track was not changed during the experiment.

The subjects were tested singly, and they did not get any further instruction. Only after the experiment were they given an opportunity to play with the track.

3. Results

The strips yielded a sequence of numbered marks for each subject. We measured the distance from the goal for each mark. When the ball rolled too far the result was recorded as positive, otherwise it was negative. The results could then be studied by diagrams which revealed each subject's strategy.

We present here some typical cases, to show different abilities of mastering the problem and various strategies.

(1) *Ert.* (6 years)

This boy always put the ball at the top of the track. It then rolled off the end of the track. Hints to the child that he could place the ball elsewhere did not change his behaviour. The child then started crying. The test was stopped and the child was comforted.

The behaviour of this child makes clear that he did not yet recognize the existence of a connection between the starting and stopping points. He did not discover this fact during the experiment.

(2) *Teu.* (14 years)

This boy also very often placed the ball at the top, but sometimes he preferred another point. He seemed to choose one extreme starting point after the other

(Figure 2).

Protocol

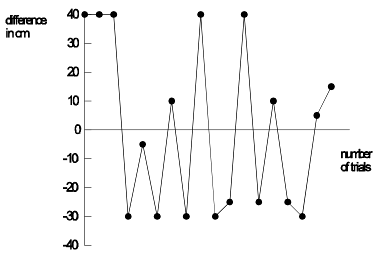

Fig.2. *Teu*. (14 years)

This child's procedure for placing the ball was not as fixed as that of the previous subject. A learning process might have started. But the relationship between the starting point and the stopping point had not yet been discovered.

(3) *Wey*. (6 years)

This boy began at the top of the track, but then moved to other points (Figure 3). The starting points were chosen without a guiding principle. The boy learnt that for any starting point there exists only one stopping point. But he did not find a way of discovering the correct starting point systematically.

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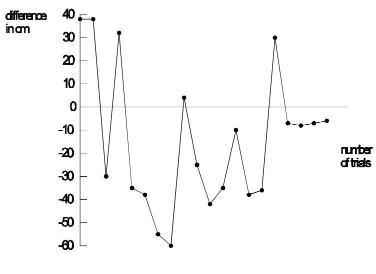

Fig. 3. *Wey*. (6 years).

(4) *Elm.* (14 years)

This boy placed the ball systematically (Figure 4), but obviously he took the wrong direction again and again. He seemed to think that the further left the ball starts, the further to the left it will stop. He assumed a somewhat rigid connection between the two points. Although the experiment informed the subject about the wrong result, he was not yet able to find a converging strategy.

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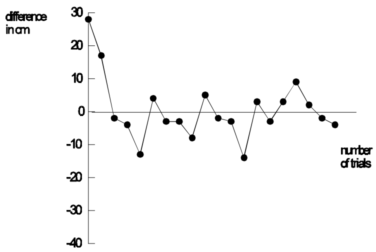

Fig. 4. *Elm*. (14 years).

(5) *And.* (10 years)

This girl moved on one occasion in the wrong direction (Figure 5), but quickly learnt from the wrong result and thereafter used a successful strategy. She knew that there was a relationship between the two points and she assumed that the more to the left the ball started the more to the left it stopped. But she revised this assumption and found out that the higher the ball starts the further it rolls. She then approximated to the goal step by step.

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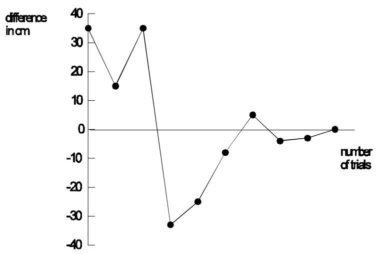

Fig. 5. And. (10 years)

(6) *Sve.* (12 years)

Figure 6 shows that this boy never took the wrong direction. He used a search strategy by approximating the correct starting point from both sides. He knew that there exists only one stopping point for every starting point, and that the higher the ball starts the further it rolls.

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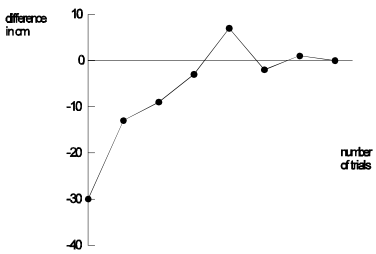

Fig. 6. Sve. (12 years).

(7) *Vol.* (14 years)

This boy also placed the ball systematically, but he carefully approached the correct starting place from one side only (Figure 7). He too had the same understanding of the process as the previous subject but he used a different strategy.

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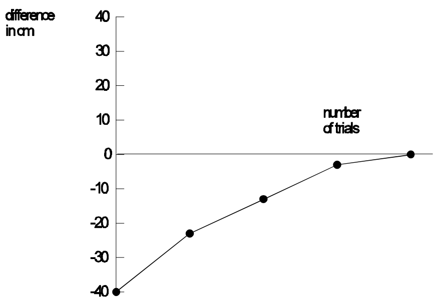

Fig. 7. Vol. (14 years).

From these results we can identify the following stages:

Stage 0: The subject always starts from the same starting point, and expects that the ball will stop at the goal on any occasion by chance. Bad results do not change the behaviour.

Stage 1: The subject starts at different points, but without a system. It is aware of a relationship between starting and stopping points, but it is convinced that the correct point can only be found by trial. Bad results do not change the behaviour.

Stage 2: The subject places the ball systematically. It knows that the higher the ball starts the further it rolls.

The transitions between the stages represent phases of learning by trial and error. By Stage 1-2 we denote a phase in which the subject starts with the knowledge of Stage 1 but learns from mistakes, and arrives at the knowledge of Stage 2.

Among the converging strategies we found two types:

Type A: Oscillating: the goal is approached from both sides.

Type B: Approximating: the goal is approached from one side only.

To study the influence of age we start with the distribution of age in our sample:

Table I

Distribution of age

age	4	5	6	7	8	9	10	11	12	13	14	15
number	1	0	2	2	4	7	2	5	9	12	13	2

Because our main interest was directed towards the discovery of the monotonic property, we preferred children of the age group 10-14. We also wanted to avoid discouraging situations for younger children. Referring to PIAGET, we expected the transition from Stage 1 to Stage 2 in the age group 11-12. So we presumed that there would be a difference between the behaviour of age group 4-11 and age group 12-15.

Table II shows the relationship between age and success.

Table II

Comparison of age and success

number	age		
of trials	4-11	12-15	total
1-8	9	24	33
9-20	15	12	27
total	24	36	60

FISHER's test yielded a significant effect ($p = .03$) showing that efficiency improves with age.

The relationship between age and stage can be seen from Table III.

Table III

Comparison of age and stage

stage	age		
	4-11	12-15	total
1 or lower	8	4	12
1-2	8	10	18
2	8	22	30
total	24	36	60

The chi-square test shows a significant dependence between stage and age ($\chi^2 = 5.93$; $p<0.05$).

Finally we looked for a relationship between age and type of strategy. We considered only the successful Ss, who worked systematically (Table IV).

Table IV

Comparison of age and strategy

strategy	age		
type	4-11	12-15	total
oscill.	16	21	37
approx.	2	11	13
total	18	32	50

By Fisher's test one can see that the proportion of approximaters increases with age ($p = 0.07$).

An influence of gender on success and strategy could not be found. Take for example the relationship between gender and success as shown in Table V (0:

no success; 1: successful).

Table V

Comparison of gender and success

success	gender		
	female	male	total
0	3	7	10
1	19	31	50
total	22	38	60

Other observations of interest were the following. Our subjects were interested in playing with the ball. Most of them asked us if they could repeat the game. Many of them then changed the conditions, e.g. the gradient of the track, and studied the influence of these changes. No subject used the pencil to mark the starting point. They all kept it in mind by their visual memory. During the experiment we could observe some older children who tried to measure the distances with their fingers, and they tried to make a rough estimate to find a solution. They presumed proportionality, and their findings did not contradict their assumption.

Finally we observed some children who were rather successful but before reaching the goal they went back to their first starting position and tried to find the goal in a new experiment.

4. Discussion

We start by discussing the observations relating to the aspect *of functional thinking.*

Younger children such as *Ert*. (1) are not aware of a dependence between starting and stopping point. They keep one quantity constant and expect that the other quantity will change by chance. Bad results do not help to discover the dependence.

A first attempt to overcome this limitation is the search for other convenient positions, but still just a few positions are preferred (*Teu*.).

A first appearance of functional thinking can be seen when children start to change one variable and expect changes in the other variable. Because the assumption of the dependence is not false it does not need to be revised. But because of the limitation of trials the chance of success is rather small.

A systematic procedure of the children reveals additional assumptions, which can be wrong without being revised (*Elm*.) or else they lead to the discovery of the correct monotonic property (*And*.).

Finally there are subjects such as *Sve*. and *Vol*. who are aware of the dependence and assume a correct monotonic property.

The stages observed can be interpreted as stages of functional thinking. Let us assume an aiming problem in which the relationship between two variables x and y is a monotonic function.

Stage 0: No correlation is seen. The functional dependence is not discovered.

Stage 1: A functional relationship is known to exist. But variations of one variable do not lead to the discovery of the monotonic property.

Stage 2: The monotonic property of the functional dependence is recognized.

The interim stages are characterized by the ability to make further assumptions which are not convenient but can be revised so that the knowledge of the next stage can be gained.

The ability to discover the monotonic property develops at 11-12 years. But we found a 14 year old child at Stage 0 and a 7 year old child at Stage 2.

We did not offer instruction to those who did not succeed, therefore we do not know whether they could have learn by instruction.

The reason that successful subjects start with the correct assumptions was, we assume, due to relevant experiences gained in equivalent situations. For our aiming problem such experiences can be gained by playing with toy cars, with a model railway, by sledding, by cycling, or by experiences from physics instruction.

PIAGET described the development towards the discovery of proportionality in three stages: At the first stage children are not aware of a dependence. Solutions arc sought by guessing. During the second stage the children are aware of a bijective dependence. Solutions are sought by estimation or later by calculation on the basis of additive changes. They assume that additive changes of one quantity lead to the same additive changes of the other quantity. Proportionality is discovered in the third stage and used for correct calculations which solve the problem.

The estimations in the second phase can be interpreted as the result of the assumption of a monotonic property. But this is not an important step in the context of his experiments. On the other hand we observed some older children who tried to make rough estimates on the basis of proportionality. They felt that a calculation would lead most quickly to a solution.

So there is obviously a correspondence between our sequence of stages and PIAGET's. But the weight and the type of assumptions during the intermediate stages differ corresponding to the problem type.

Piaget found that for inverse proportional dependences older subjects often started with the assumption of proportionality. This is confirmed by SUAREZ (1977). It is known from mathematics instruction that this can be avoided if the problem solvers are trained to first test whether the dependence is monotonic increasing or decreasing (KIRSCH 1969). Therefore the monotonic property is also of importance in problems solved by proportional or inverse proportional methods when both types are offered to the students. Physical tasks have been criticized on the basis of their requirement of physical knowledge (KARPLUS *et al.* 1983). But only prephysical experience is helpful for mastering our task. On the other hand, in our opinion, experiences like these are responsible for developing functional thinking.

Our investigation can also be discussed under the aspect of *search strategies.* It is known from research on search problems that efficiency and systematic behaviour improves with age (DROZ and POTTER 1969). This is confirmed by our investigation; furthermore, it explains this effect for search processes in which the result of a trial is uniquely determined by a function:

> The efficiency in goal seeking improves through the development of functional thinking.

Some observations cannot sufficiently be explained with the help of functional thinking. The discovery of the monotonic property leads to different successful strategies. Approximating is a strategy by which the changes appear in one direction. Oscillating is a procedure by which the aim is encircled.

These strategies are known from other aiming problems. MEISSNER (1985a) observed students playing a calculator game in which they had to guess a number, such that multiplication with a certain number would lead to a result which hit a given interval. MEISSNER could identify the two strategies, but also some substrategies (e.g. "speed up", "slow down"). He did not study dependence on age or repetition.

Our investigation shows that approximating increases with age. The concern for minimization may be the reason for the increase: The more careful older subjects wanted to avoid unnecessary mistakes.

The different strategies possibly result from different cognitive styles. Approximating as a "careful" procedure would be preferred by *reflective* subjects, whereas oscillating is a more "courageous" procedure prefered by *impulsive* subjects (KAGAN 1965). But this assumption does not help us to understand the age dependency of the chosen strategy.

The choice of the strategy can also be the result of experiences gained by actions. The two strategies are general strategies in search processes (e.g. sighting, focusing, and steering). Mistakes can be explained by rough corrections which are typical for inexperienced people in steering situations.

Ceasing and starting again at an earlier position is a rather useless procedure for iterations. But it is a behaviour which seems to be typical for trial situations. MEISSNER (1985b) interpreted this as a "substitute action" to remain active when a successful procedure is not yet known. I prefer to understand it as a behaviour which is trained in school: If you are lost try it again. And there is another experience from problem solving in mathematics: It is easier to avoid a mistake in a new attempt than to find a mistake in series of calculations. Perhaps it would be easier to explain these effects if one knew the subjects' behaviour under repetition.

No subject used the pencil for marking. This can be explained by the subjects' ability to remember a position by their visual memory. It is known from the puzzle "Memory" that children are very strong in remembering objects in special positions. The veining of the wooden track may have been an aid for orientation as well. Perhaps they also could remember the position from the proportion of the sections on the ramp.

5. Implications for mathematics education

In Germany a spiral approach to the concept of function starts at the age of 13. One can assume that most of the students of this age arc able to understand fundamental properties of functions in situations from their environment. But the teacher must be careful: we had a 14-year-old subject at stage 0! Mathematics instruction should offer experiments to the students from which they can gain experience. Problems should lead to conjectures, and they should get the opportunity to check them (VOLLRATH 1978).

Approximations and iterations play an important role in modern mathematics, especially with the increasing use of computers. Iteration was considered a suitable method by most of the subjects in our investigation. On the other hand, many people consider approximations to be somewhat unmathematical: mathematics has to be exact! It is important to make clear that the development of a converging search strategy is the result of mathematical understanding, and very often of a mathematical idea. The development of functional thinking can improve the ability to find search strategies; and, vice versa, looking for search strategies can improve functional thinking. We therefore suggest putting more emphasis on search problems in mathematics instruction.

References

Andelfinger, B., Didaktischer Informationsdienst Mathematik. Thema: Proportion, Landesinstitut für Curriculumentwicklung, Lehrerfortbildung und Weiterbildung, Neuß 1981.

Burton, L., Mathematical thinking: The struggle for meaning, Journal for Research in Mathematics Education 15 (1984), 3549.

Case, R., Intellectual development and instruction: A neo-Piagetian view, in A. Lawson (ed), The Psychology of Teaching for Thinking and Creativity, ERIC, Columbus, Ohio 1979.

Droz, R. and Potter, T. F., Search and exploration: Modification of efficiency and systematicity with age and repetition, Archives de psychologie 40 (1969), 25-35.

Freudenthal, H., Didactical Phenomenology of Mathematical Structures, Dordrecht (D. Reidel) 1983.

Gutzmer, A., Die Tätigkeit der Unterrichrskommission der Gesellschalt Deutscher Naturforscher und Ärzte, Leipzig (Teubner) 1908.

Häussler, P., Denken und Lernen Jugendlicher beim Erkennen funktionaler Beziehungen, Bern (Huber) 1981.

Hart, K. M., Children's Understanding of Mathematics: 11-16, London (Murray) 1981.

Inhelder, B. and Piaget, J., The Growth of Logical Thinking from Childhood to Adolescence, Basic Books, New York 1958.

Kagan, J., Impulsive and reflective children: Significance of conceptual tempo, in J. D. Krumboltz (ed.), Learning and the Educational Process, Chicago (Rand McNally) 1965.

Karplus, R., Pulos, S. and Stage, E. K., Proportional reasoning of early adolescents, in R. Lesh and M. Landau (eds.), Acquisition of Mathematics Concepts and Processes, New York (Academic

Press) 1983.

Kirsch, A., Eine Analyse der sogenannten Schlußrechnung, Mathematisch-Physikalische Semesterberichte 16 (1969), 41-55.

Meißner, H., Selfdeveloping strategies with a calculator garne, in L. Streefland (ed.), Proceedings of the Ninth International Conference for the Psychology of Mathematics Education, Vol. 1, PME, Utrecht 1985a.

Meißner, H., Versuchen und Probieren – Beobachtungen zum mathematischen Lernprozeß, in W. Dörfler and R. Fischer (eds.), Empirische Untersuchungen zum Lehren und Lernen von Mathematik, Wien (Hölder-Pichlcr-Tempsky) 1985a.

Piaget, J., Grize, J. B., Szeminska, A., and Bang, V., Epistémologie et psychologie de la fonction, Paris (Presses Universitaires de France) 1968.

Ricco, G., Les premieres acquisitions de la notion de fonction linéaire chez l'enfant de 7 à 11 ans, Educational Studies in Mathematics 13 (1982), 289-327.

Siegler, R. S., Three aspects of cognitive development, Cognititve Psychology, 8 (1976), 481-520.

Suarez, A., Formales Denken und Funktinosbegriff bei Jugendlichen, Bern (Huber) 1977.

Tourniaire, F. and Pulos, S., Proportional reasoning: A review of tbe literature, Educational Studies in Mathematics 16 (1985), 181-204.

Vollrath, H.-J., Schülerversuche zum Funktionsbegriff, Der Mathematikunterricht 24 (4) (1978), 90-101.

Vollrath, H.-J., Methodik des Begriffslehrens im Mathematikunterricht, Stuttgart (Klett) 1984.