[100] Towards an authentic teaching of mathematics – Hans-Georg Steiner's contribution to the reform of mathematics teaching

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Abstract: *Hans-Georg Steiner was the "motor of the reform" of mathematics education in Germany. His main concern was to promote authentic teaching. His suggestions for teaching mathematical structures stimulated the process of reform, but were criticised as well. Two controversies are studied in this paper. The controversy with Detlef Laugwitz in 1965 was about the dichotomy "axiomatics vs. constructiveness". Another controversy with Alexander Wittenberg in 1964 was about the problem of "elementary". The following considerations can show the need for fundamental didactical analyses in mathematics education as they were initiated by Hans-Georg Steiner.*

1. About the authenticity crisis of mathematics teaching

During the sixties a worldwide crisis was approaching mathematics teaching. University teachers were criticizing the insufficient mathematical education of their students who had no idea of modern mathematics. Especially young teachers at the *Gymnasium* in Germany felt that the mathematics they were expected to teach was old-fashioned in comparison with the mathematics that they had been taught at the university. Hans-Georg Steiner (1928–2004) described the situation in 1959 as follows:

> „Der Fortschritt der Mathematik in den letzten hundert Jahren hat nicht nur durch den Reichtum an neuen mathematischen Theorien und Einzelergebnissen, sondern auch durch die Vertiefung der begrifflichen Grundlagen und die Entwicklung von weiterreichenden Verfahrensweisen ganz erheblich über jene Mathematik hinausgeführt, die dem Unterricht an den Gymnasien in der ersten Hälfte unseres Jahrhunderts zugrundegelegen hat, eine Mathematik,

> die im wesentlichen schon um 1800 vorlag. Das hat im Laufe der letzten Dezennien, wo das Fortschreiten der Wissenschaft in die Breite und in die Tiefe immer mächtiger wurde, die Schule aber relativ stark an traditionellen Stoffen und Auffassungsweisen festhielt, zu einer solchen Rückständigkeit des mathematischen Unterrichts geführt, daß in Deutschland wie in vielen anderen Ländern, in denen die Situation bei aller Unterschiedlichkeit der Schulformen ganz ähnlich ist, eine große Beunruhigung eintreten mußte." (Steiner 1959, p. 5)
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> *(The progress of mathematics in the last one hundred years has significantly led beyond what was the basis for education at secondary schools in the first half of this century (which was a kind of mathematics that was essentially already available around 1800) not only through the abundance of new mathematical theories and particular theorems, but also by a deepening of the conceptual fundamentals and the development of further reaching approaches. This has led to a substantial backwardness in mathematics education during the last decades when mathematical science made ever greater advances both in width and in depth, while the schools quite strongly adhered to traditional subject matters and interpretations. This backwardness was such that it has led to great concern in Germany and in many other countries where the situation is similar, in spite of differences in educational structures.)*

Three aspects were significant to him for the development of modern mathematics:

- *Characterizing axiomatics:* Under the influence of David Hilbert's (1862–1943) *Grundlagen der Geometrie* (1899) a renaissance of axiomatics had taken place in mathematics. Modern axiomatics has given up to define the fundamental concepts as it was common since Euclid's (c. 300 B. C.) *Elements* and defines them implicitly by a system of axioms. The goal is a theory with isomorphic models.

- *Abstracting axiomatics:* Bartel Leendert van der Waerden's (1903–1996) *Moderne Algebra* (1930) had a different axiomatic approach. The algebraic structures such as group, ring, and field were axiomatically defined by common properties of different models.
- *Formalizing:* Georg Cantor's (1845–1918) Mengenlehre (*set theory*) und Gottlob Frege's (1848–1925) *Begriffsschrift* (*Concept Script*) (1879) lead to a critical attitude towards the language of mathematics. Consequently, formal logic and set theory were the reasons of an intensive study of the fundamentals of mathematics.

Mathematics education has the goal to teach students fundamental contents, to develop mathematical thinking, and to let them know, how mathematics helps to gain knowledge and to solve problems. Thus mathematics teaching has to give answers to three fundamental questions:

- What is mathematics?
- How does mathematics appear?
- What is the use of mathematics?

The first question is directed to mathematical *content*, the second to mathematical *method*, and the third to *applications* of mathematics. The answers have to take into account that students get the chance to experience mathematics in its different aspects as Alexander Wittenberg (1925–1965) insisted:

> „Im Unterricht muß sich für den Schüler eine *gültige Begegnung* mit der Mathematik, mit deren Tragweite, mit deren Beziehungsreichtum, vollziehen; es muß ihm am Elementaren ein echtes Erlebnis dieser Wissenschaft erschlossen werden. Der Unterricht muß dem gerecht werden, *was Mathematik wirklich ist*." (Wittenberg 1963[1], pp. 50-51)
>
> *(During classes, the student must accomplish a valid encounter with mathematics, with its significance and its evocativeness; a real*

> *experience must be opened to him through the elementary. Classes must do justice to what mathematics really is.)*

He speaks of a “valid encounter” of mathematics as it “really” is. Only under these conditions can mathematics education contribute specifically to general education. Mathematics education that helps students to gain valid experience of mathematics is called *authentic* (Vollrath 2001). Referring to Wittenberg mathematics has therefore to be taught *genetically*. Steiner agreed and demanded: mathematics has to be taught in the same way “as the productive mathematical work proceeds.” Therefore “mathematics instruction has to reach the modern axiomatic point of view.” (Steiner 1959, p. 14).

From Steiner’s view mathematics education at the *Gymnasium* (secondary school that qualifies pupils for studies at a university or equivalent institutions) offered an antiquated impression of mathematics. His paper *Das moderne mathematische Denken und die Schulmathematik* (*Modern mathematical thinking and school mathematics*) that appeared in the journal “Der Mathematikunterricht“ from 1959 was first presented in 1957 at the “23. Pfingsttagung zur Pflege des Zusammenhangs von Universität und Schule“ in Münster. It resulted from his experiences as *Referendar* (teacher candidate) of the *Studienseminar* (Institute for in-service education) in Münster from 1955 to 1957.

After two years as scientific assistant of Heinrich Behnke (1898–1979) at the University of Münster and four years as *Gymnasium* teacher in Münster Hans-Georg Steiner went back to university. In 1963 he became “Studienrat im Hochschuldienst” at the University of Münster. There he could develop his didactical ideas; moreover, he enjoyed international contacts, and he became the “motor of the reform” of mathematics education in Germany (Griesel 1988).

He formulated his program for the reform of the mathematics education at the *Gymnasium* in his paper *Menge, Struktur, Abbildung als Leitbegriffe für den modernen mathematischen Unterricht* (*Sets, structures, and mappings als* *leading concepts in mathematics education*) (Steiner 1965a). It resulted from discussions about the syllabus for mathematics education in Nordrhein-Westfalen and for the “Rahmenplan” of the “Deutscher Verein zur Förderung des mathematischen und naturwissenschaftlichen Unterrichts” (MNU), in which Steiner was strongly involved.

Hans-Georg Steiner (1928–2004)

2. The program of the reform

The “Rahmenplan” (national curriculum framework) was intended to be the model for the future syllabi of the “Länder” (the federal states) of the Federal Republic of Germany. It was presented at the annual meeting of the MNU in 1965 that took place in Nürnberg. This syllabus was obviously influenced by Steiner’s ideas. It said in the introduction:

> „Die neuere Entwicklung der mathematischen Wissenschaft hat vereinheitlichende Vorstellungen hervorgebracht, die den inneren Zusammenhang zwischen den verschiedenen mathematischen Teilgebieten hervortreten und auch die Beziehungen zu den Anwendungen prinzipiell verständlich werden lassen. Die Mathematik beschäftigt sich mit strukturierten Gebilden, d. h. mit

> bestimmten Mengen von Objekten und den auf ihnen definierten Strukturen, mit der gedanklichen Konstruktion neuer Gebilde und mit Zusammenhängen zwischen Gebilden, insbesondere mit Zuordnungen (Relationen und Funktionen bzw. Abbildungen). Die Gebilde selber erscheinen dabei als Modelle für abstrakte Strukturen, die durch entprechende Axiomensysteme beschrieben und mit logischen Hilfsmitteln begrifflich beherrscht werden.
>
> Diese Auffassung der wissenschaftlichen Mathematik gibt auch dem mathematischen Schulunterricht ein neues Verhältnis zu seinem Inhalt. Auch in ihm kommt es wesentlich darauf an, die jeweils zugrunde liegenden Objektbereiche als Mengen zu erfassen, auf ihnen gegebene Strukturen herauszuarbeiten, den Aufbau der Mathematik als gedankliche Konstruktion von Gebilden aus vorgegebenen einfacheren zu verstehen und die nötigen logischen Hilfsmittel zu entwickeln.“ (MNU 1965, p. 3)
>
> *(The newer development of mathematical science has given rise to unified views that made in principal the internal connection between the different mathematical areas and also the relationship to the applications comprehensible. Mathematics deals with structured entities, i.e., with certain sets of objects and structures defined on them; it deals with the theoretical construction of new entities and with relationships between entities, particularly with assignments (relations, functions and mappings, resp.). The entities themselves appear as models of abstract structures that are described by appropriate axiomatic systems and are conceptually mastered by means of logical tools.*
>
> *This perception of the mathematical science also gives the teaching of mathematics in schools a new relationship with its content. It also essentially requires to consider the underlying object areas as sets, to work out given structures on them, to understand the composition of mathematics as a theoretical construction of entities out of given simpler ones, and to develop the necessary logical tools.)*

The “Rahmenplan” then gave hints to sets, structures, mappings (functions), and logical concepts as contents of teaching. The syllabus suggested a way of developing structural thinking.

In grade 5 and 6 sets of concrete objects should be studied. Natural numbers and fractional numbers were considered as sets of numbers. In geometry sets of points, mappings and their compositions should be studied.

In grade 7 to 10 the number system was expanded. The teaching of equations and inequalities was strongly influenced by ideas of set theory and logic. Functions should be introduced as special relations. Geometry had to be developed with the aid of mappings. Sets of numbers and their operations had to be interpreted as algebraic structures using the concepts of group, ring and field.

One objective in grade 11 to 13 was: mathematics education "has to contribute to axiomatic thinking and has to deal with problems of the foundation of mathematics in at least one limited topic." The given examples could contribute to characterizing and to abstracting axiomatics. As examples for convenient topics "characterizing the real number field through fundamental properties and introductions to groups, rings, and fields" were given. Thus, axiomatic thinking had to be developed, starting with concrete models and ending up with an example of a theory.

Obviously, the "Rahmenplan" tried to influence mathematics teaching at the *Gymnasium* with respect to *contents* through the introduction of modern concepts, and with respect to *methods* through stressing logical relations. The study of structures had to be developed beginning with the construction and study of concrete object areas. From there the students could gain experience with axiomatics.

3. The problem of balance

The "Rahmenplan" was following the international trend of modernising mathematics education and it was in line with Steiner's ideas. But a presentation of Detlef Laugwitz (1932–2000) on April 13th, 1965 about *Sinn und Grenzen der axiomatischen Methode* (*The use and the limitations of the axiomatical method*) at the meeting in Nürnberg questioned the syllabus (Laugwitz 1966).

As a professor of mathematics at the "Technische Hochschule Darmstadt" Laugwitz saw mathematics represented in an unbalanced way in the "Rahmenplan". He did not like the direction of it. In fact, he too saw the necessity of reforming mathematics education at the *Gymnasium*.

Detlef Laugwitz (1932–2000)

He complained about the lack of understanding fundamental facts, necessary abilities and motivation for dealing with mathematical problems. Therefore, he vigorously opposed the view of the "Rahmenplan":

> „Bei den Diskussionen um die Modernisierung des Mathematikunterrichts spielt gegenwärtig das Schlagwort Axiomatik eine besondere Rolle. Es wird vielfach davon ausgegangen, daß Mathematik, und zwar besonders die sogenannte moderne Mathematik, ihrer Methode nach axiomatisch sei. Auch die Materie der modernen Mathematik sieht man im axiomatischen Bereich: Man faßt die Mathematik auf als eine Wissenschaft von denkmöglichen Strukturen, anders und vielleicht besser ausgedrückt als Lehre von den formalen Systemen. ...
>
> Meine These ist: Die Voraussetzung ist falsch, und zwar sowohl, was die Bedeutung der gegenwärtig florierenden Axiomatik im Gesamtbild der Mathematik unserer Zeit betrifft, als auch im Hinblick auf die Sachverhalte, mit denen Mathematik es heute und in der absehbaren Zukunft zu tun hat." (Laugwitz 1966, pp. 16-17)
>
> *(The catchword "axiomatics" currently plays a special role in the discussions about the modernisation of teaching mathematics in school. It is often assumed that mathematics - and especially the so-called modern mathematics, is axiomatic by its method. The matter of modern mathematics is also seen in the axiomatic domain: One construes mathematics as a science of thinkable structures,*

> *differently and probably better expressed as being the theory of formal systems. ...*
>
> *My thesis is: This assumption is wrong, namely concerning the significance of the currently booming axiomatics in the overall picture of today's mathematics as well as in view of the circumstances that mathematics has to deal with today and in the foreseeable future.)*

Laugwitz got roaring applause by most of the teachers, because they were resistant to reform mathematics teaching. Later on Franz Denk (1897–1977) wrote a letter to Steiner:

> „Die Studienräte hören es gerne, wenn jemand gegen die Reformen spricht, weil das Ihre Ruhe fördert.“ (Denk to Steiner on August 6th, 1965.)
>
> *(The teachers like to hear it when someone speaks out against the reforms, because it benefits their tranquility.)*

A strong controversy between Steiner and Laugwitz happened just after the speech. Laugwitz wanted to make clear that mathematics was not the “doctrine of formal systems”. His fear was that the result of the “Rahmenplan” would be an unbalanced understanding of mathematics. This hit the reformers heavily because they just wanted to overcome the unbalanced traditional mathematics teaching which was too procedural. So they opposed strongly.

The controversy had a long history. The organizers of the annual MNU-Meeting in Aachen in 1964 had already emphasised the necessity of a reform of mathematics teaching. Laugwitz had read about it on April 14th, 1964 in the local newspaper “Darmstädter Echo” (Steiner 1965b). The paper had reported about some progress in mathematical research that resulted from growing abstraction and axiomatics. The keyword for Laugwitz was “axiomatization” that “alarmed” him. He was in fear that mathematics education was to be reformed with respect to this keyword. Therefore, he responded with his paper *Der Streit um die Methode in der modernen Mathematik* (*The struggle about methods in modern mathematics*) which he presented at the “Philosophisch-Naturwissenschaftliches Kolloquium” at the Technische Hochschule Darmstadt on May 13th, 1964. It was printed in early 1965 in the journal “Neue Sammlung”.

This paper suffered from unclear concepts, problematic examples and a lot of polemics. It was obviously written in excitement. Steiner responded with a detailed statement for the “Neue Sammlung”. But the journal refused to publish it referring to the limited understanding of its readers. It finally appeared in the “Mathematisch-Physikalische Semesterberichte” of 1965 (Steiner 1965b). Laugwitz’s presentation in Nürnberg followed the presentation in Darmstadt in his line of argumentation, only with fewer polemics.

The emotional controversy in Nürnberg was replaced by a written discussion in the journal “Der Mathematikunterricht” (MU) with the title: *Die axiomatische Methode im Schulunterricht* (1966). An article of Helmut Coers analysed the problem of existence in mathematics and made clear that Laugwitz had to formulate his positions more precisely (Coers 1966). As a result, Laugwitz published a revised version of his paper in the same number of that journal (Laugwitz 1966) from which the following quotations are taken.

4. A false dichotomy: Axiomatics vs. Constructiveness

In his paper Laugwitz, critically, analysed the axiomatic method. To him the struggle between formalism and constructivism about the foundations of mathematics had not been decided yet. He referred to recent contributions of Paul Lorenzen (1915–1994) about a constructive foundation of the number system which questioned *characterizing axiomatics.* Furthermore, he saw the danger that characterizing axiomatics in mathematics teaching would give the wrong impression that mathematics is a “closed science” where everything is at hand and only needs to be *discovered*. This way of teaching would neglect mathematical creativity with the possibility of inventing new fields of mathematics, as he said:

> „Es ist alles schon da, man braucht es nur zu entdecken. Eine wesentliche Komponente schöpferischer Mathematik würde vernachlässigt, wenn man diesem Fehlschluß folgte; das selbständige Finden von Mathematik, ja geradezu das Erfinden, macht unsere Wissenschaft zu einer fortschrittlichen Disziplin, die sich nicht im Wiederholen von Bekanntem erschöpft.“ (Laugwitz 1966, p. 21)
>
> *(Everything is already there, one only has to discover it. If one were to follow this false conclusion, an essential component of creative*

> *mathematics would be neglected. Independent discovery of mathematics, and actually invention, is what makes our science a progressive discipline which does not exhaust itself in repeating well-known facts.)*

Stressing the *constructive* aspect of mathematics could counterbalance this. He used the word "constructive" in a twofold meaning: On the one hand he used it in the way of Lorenzen to mark a position concerning the foundations of mathematics. On the other hand he wanted to describe the creative way in which mathematics is developed in research and in practical applications. And he missed a lot of that in mathematics education as well:

> „Natürlich ist rein deduktive Mathematik wissenschaftlich richtig, sie ist aber deshalb noch nicht didaktisch vernünftig. Zum Entdecken von bereits Vorhandenem oder Fabriziertem lassen andere Fächer genug Platz – wir Mathematiker sollten es uns nicht nehmen lassen, das Erfinden zu fördern!" (Laugwitz 1966, p. 37).
>
> *(Of course, purely deductive mathematics is scientifically correct, but that doesn't imply it is didactically reasonable. Other subjects leave enough space for rediscovering the existing or the fabricated – we mathematicians should remain at the wheel of encouraging invention.)*

Concerning the *abstracting axiomatics* Laugwitz admitted that it could be a fruitful *tool* in modern mathematics:

> „Es gibt wesentliche Teile der heutigen Mathematik, wo Axiomatik vielleicht als Werkzeug eine gewisse Rolle spielt, im Prinzipiellen aber doch unwesentlich ist. Man denke nur an die elementare und die analytische Zahlentheorie, die Differentialgeometrie und die Analysis überhaupt. Hier bedienen wir uns gern und mit Erfolg gruppentheoretischer und funktionalanalytischer Hilfsmittel, aber das sind eben nur Werkzeuge; für die wirklichen Grundlagen wird damit nichts Wesentliches geleistet, und es ist zu überlegen, ob daher nicht die Methoden der abstrahierenden Axiomatik erst dann einsetzen können, wenn eine hinreichende Kenntnis konkreter Sachverhalte vermittelt worden ist." (Laugwitz 1966, p. 28.)
>
> *(There are essential parts of today's mathematics where axiomatics perhaps plays a certain role as an instrument, but is principally irrelevant. Consider, for instance, elementary and analytical number theory, differential geometry and analysis in general. Here, we*

willingly and successfully avail ourselves of utilities from group theory and functional analysis, but these are just tools. Nothing of significance is done for the real basics and one should therefore think about whether the methods of abstract axiomatics can only apply when a sufficient knowledge of concrete facts has been imparted.)

But abstracting axiomatics as an end in itself was refused by him as he also strongly emphasized in his Darmstadt paper:

> „Mein zweiter Einwand gegen den Axiomatizismus ergibt sich aus dem gegenwärtigen Zustand der Mathematik. Der Inhalt der Mathematik wird nämlich zur Zeit wesentlich von der Form, also der vorherrschenden axiomatischen Methode, geprägt. Da werden ‚Homologie'- und ‚Cohomologie'theorien axiomatisch entwickelt, und jeder kleine Topologe strebt nach dem Aufbau einer großen Theorie aus einem von ihm selbst angegebenen Axiomensystem, und das Ergebnis sind bestenfalls Sätze, die von der Mehrheit der mathematischen Zeitgenossen als Triumphe des Geistes gefeiert werden, die aber im Gesamtgebäude der Mathematik nur dann überhaupt zu sehen sind, wenn man ein winziges Eckchen der Mathematik in unendlicher Vergrößerung betrachtet. Diese harte Bemerkung kann ich durch beliebig viele Beispiele belegen." (Laugwitz 1965, p. 17)
>
> *(My second objection to axiomaticism arises from the current state of mathematics. The content of mathematics is presently substantially affected by the form, that is, the prevailing axiomatic method. Homology and cohomology theories are developed axiomatically, and every minor topologist strives for the construction of a great theory out of an axiomatic system of his own. In the best case, the results are theorems that most of the mathematical contemporaries celebrate as triumphs of the mind, whereas they are really only visible in the whole edifice of mathematics if one views a tiny corner of mathematics in infinite magnification. I can give an arbitrary number of examples for this harsh remark.)*

Therefore, he wanted to keep school free from this danger even though he admitted that topics of abstracting axiomatics could be very stimulating when taught by enthusiastic teachers:

> „daß man Gruppen, Ringe, Körper, Vektorräume einer aufgeschlossenen Oberstufe durch einen Lehrer, der mit diesen Strukturen wirklich lebt, nahebringen kann, ist nicht nur möglich, sondern sogar wünschenswert.“ (Laugwitz 1966, p. 28)
>
> *(One can make groups, rings, fields, vector spaces accessible to an open-minded senior class through a teacher who really lives with these structures; this is not only possible, but in fact desirable.)*

This was the right statement to settle the quarrel. In detail, it was possible to find an agreement at *content* when Laugwitz got the impression that the *method* was right. But just that was the problem of the “Rahmenplan”. Laugwitz was angry about the predominance of the axiomaticians.

Steiner did not see any discrepancies in the “Rahmenplan” because he just understood the syllabus as a result of the attempt to balance mathematics education. To his view the traditional mathematics teaching at the *Gymnasium* had only been constructive in a narrow sense. He wrote:

> „Der ganze traditionelle Mathematikunterricht am Gymnasium war im wesentlichen konstruktiv angelegt, ohne dabei allerdings das (natürlich nicht bis in die Einzelheiten durchgeführte) konstruktive Vorgehen als solches und in Abgrenzung zu anderem Vorgehen bewußt zu machen. Durch die Einführung der Grundvorstellungen und Methoden der sog. modernen Mathematik wird gerade versucht, über die damit dem Unterricht sachlich wie methodisch auferlegten Beschränkungen hinauszukommen.“ (Steiner 1965b, p. 19)
>
> *(The whole traditional way of teaching mathematics at the Gymnasium was essentially arranged constructively, without, however, making the students aware of the constructive approach as such (of course not carried out in all details) and of the dissociation with other approaches. By introducing the basic concepts and methods of so-called modern mathematics one is currently trying to get beyond the restrictions to content and methods in education that were thus imposed.)*

Steiner was convinced that the syllabus could offer a chance to a broader view of constructiveness through teaching the ideas of modern mathematics.

> „Andererseits gibt es jedoch keinen Grund, die traditionellen konstruktiven Ansätze nicht auch im modernen Unterricht

> beizubehalten. Ja, es besteht an vielen Stellen sogar die didaktische Notwendigkeit, konstruktive Methoden auch unmittelbar zur Einführung mathematischer Gebilde mit heranzuziehen, etwa bei der Behandlung der natürlichen Zahlen in der Unterstufe. Es ist aber in jedem Falle vorteilhaft, mehr als das konstruktiv Erreichbare zur Verfügung zu haben. So ist die Didaktik der Funktionenlehre z. B. dadurch viel durchsichtiger und erfolgreicher geworden, daß man den konstruktiven Teil unter dem Gesichtspunkt des rechnerischen Funktionsbegriffs zwar sorgfältig entwickelt – was selbstverständlich ist –, ihn darüber hinaus aber in einen größeren mengentheoretisch-begrifflichen Rahmen stellt. Der Hinweis auf die Funktionenlehre deutet zugleich an, in welchem Sinne das Konstruktive überhaupt einen natürlichen, bedeutsamen Platz innerhalb der modernen Mathematik und des Unterrichts findet, nämlich unter dem Aspekt des Algorithmischen.“ (Steiner 1965b, p. 20).
>
> *(On the other hand, there is no reason not to retain the traditional constructive approaches in modern school teaching. And yes, there are many situations where it is even a didactical necessity to immediately bring up constructive methods for the introduction of mathematical entities, for instance when treating natural numbers in the lower grades. But it is in any case advantageous to have available more than what is constructively achievable. For example, the didactics of the study of functions has indeed become much clearer and more successful by not only carefully developing the constructive part in view of the calculations - which goes without saying -, but furthermore by placing it in a larger set-theoretical framework. The pointer to functions indicates in which sense the constructive finds its natural, significant position within modern mathematics and education in the first place, namely under the aspect of the algorithmics.)*

Steiner too did not restrict “constructive” to Lorenzen’s understanding; he used this word more in the sense of *algorithmic*. Therefore, he was convinced that the constructiveness could be promoted by axiomatics. Steiner made clear that Laugwitz had started a discussion about a “false dichotomy” (Hilton 1976). But Laugwitz would not agree. Finally, they had different ideas about the concept of constructiveness.

The discussion between Laugwitz and Steiner came to an end with Steiner's paper *Mathematisierung und Axiomatisierung einer politischen Struktur* in the same issue of the "Mathematikunterricht" (Steiner 1966b). Steiner wanted to show that structural thinking can contribute to both, applied mathematics as well as to pure mathematics. He reported that he once – as a teacher at the *Gymnasium* – had been stimulated by a book from John G. Kemeny, James L. Snell und Gerald L. Thompson (Kemeny, Snell, Thompson 1957) to develop a teaching sequence about voting-bodies for 13^{th} grade students. It is not known whether Steiner could convince Laugwitz with that example. But Laugwitz invited him to write a dissertation about that topic. Steiner finished it in 1969 (Steiner 1969a), and earned his Ph.D (Dr. rer. nat.) at Darmstadt. Thus, the conflict between the two "fighters" for fruitful mathematics teaching at the *Gymnasium* came to a happy end in a really constructive way!

The conflict shows the difficulties to get a valid view of mathematics from experts. The consensus of the experts in the beginning became illusory by the end of the reform process. Especially, advisers can have a rather restricted view. The end of the so called "New Math" came when prominent professors of mathematics became strictly critical as they saw the results of New Math practiced on their own children and students (e. g. Kline 1973). This development made clear that the reformers had underestimated the problem of the representativeness of the consultants.

5. The problem of "elementary"

With his paper on voting-bodies Steiner tried to convince still another critic. And this conflict, too, had to do with structural concepts in mathematics instruction but different arguments were used.

Wittenberg had very critically written about structural concepts with respect to mathematics teaching at the *Gymnasium* in his book *Bildung und Mathematik* (1963[1]):

> „In der höheren Mathematik werden jene Begriffe und Methoden nicht um ihrer selbst willen eingeführt, sondern weil sie mathematisch etwas leisten – sie dienen dazu, neuartige mathematische Erkenntnisse zu erschließen. Wäre dem nicht so, so würde sich kein schöpferischer Mathematiker dazu hergeben, auch nur einen Gedanken an sie zu verschwenden. Am Gymnasium

leisten sie aber charakteristischerweise nichts. Sie bleiben Selbstzweck – und damit Unsinn...“ (Wittenberg 1963[1], p. 55).

(In higher mathematics, those terms and methods are not introduced for their own sake, but rather because they contribute something mathematically - they serve to open up new mathematical insights. If it wasn't like that, no creative mathematician would have anything to do with wasting even one thought on them. At the Gymnasium, however, such terms typically do not contribute anything. They remain an end in themselves - and therefore nonsense...)

Alexander Israel Wittenberg (1925–1965)

Wittenberg refused teaching those concepts because he found them useless for *Gymnasium* students. Steiner’s reaction in a paper at the MNU-meeting 1964 in Aachen revealed his own standpoint:

„Dies ist eine rein dogmatisch vorgetragene Behauptung, für deren Gültigkeit uns WITTENBERG bisher den Nachweis schuldig geblieben ist. Ihr stellen wir strikt die These entgegen, daß die neue mathematische Denkweise und moderne Theorien, Begriffe und Methoden in einem gewissen Ausmaß charakteristischerweise gerade am Gymnasium etwas leisten.“ (Steiner 1964/1965 , p. 194).

(This is a purely dogmatically stated claim for which WITTENBERG did not give any proof of validity so far. We counter with the strict proposition that the new mathematical way of thinking and modern

> *theories, terms and methods typically contribute something especially at the Gymnasium.)*

He pointed out that just those concepts were very useful for *Gymnasium* students.

Wittenberg and Steiner had an intensive discourse. Steiner referred to his paper on voting-bodies that he considered as an example for a profitable application of structural concepts. They had a personal talk at a meeting in Utrecht in December 1964. After that he was convinced that Wittenberg had agreed.

“The axiomatization of a political structure” remained a key example for Steiner that he very often referred to (z. B. Steiner 1969a, 1975, 1976).

When Steiner and Wittenberg discussed the *use* of structural concepts for *Gymnasium* students it was about the acquisition of a new mathematical understanding. The crucial point for Wittenberg was that the students could gain this understanding within the scope of their experience. In this case Wittenberg spoke of *valid experience*. He wrote:

> „Von ‚gültiger Erfahrung‘ der Mathematik kann denn auch nur in dem Maße die Rede sein, wie der Unterricht nicht nur die Ergebnisse, sondern das ganze Vorgehen in überzeugender Weise innerhalb des geistigen Erfahrungsbereichs des Schülers zustandekommen läßt.“ (Wittenberg 1963[1], p. 59).
>
> *(We can only talk about 'valid experience' of mathematics in as much as teaching in school does not only allow the results but the whole approach to come about convincingly within the student's intellectual set of experiences.)*

He was convinced that only dealing with *elementary* topics is convenient for gaining valid experience.

> „Im Unterricht muß sich für den Schüler eine *gültige Begegnung* mit der Mathematik, mit deren Tragweite, mit deren Beziehungsreichtum, vollziehen; es muß ihm am Elementaren ein echtes Erlebnis dieser Wissenschaft erschlossen werden. Der

> Unterricht muß dem gerecht werden, *was Mathematik wirklich ist*.“ (Wittenberg 1963[1], pp. 50-51)
>
> *(During classes, the student must accomplish a valid encounter with mathematics, with its significance and its evocativeness; a real experience must be opened to him through the elementary. Classes must do justice to what mathematics really is.)*

His concept of “elementary” referred to *the accessibility* of the content for students.

In a similar way, “elementary” is seen in science where it is common to call topics, concepts, problems, and methods that are directly accessible elementary. This is e.g. elementary geometry. It is limited to such concepts as points, straight lines, angles, simple polygons, circles, planes, and fundamental solids such as cubes and spheres. Higher geometry e.g. deals with curves and more complicated solids. Obviously, it is difficult to distinguish between these areas; e.g. one can question if conic sections belong to elementary or to higher geometry (Vollrath 2001, pp. 38-42).

Elementary geometry can also be distinguished from higher geometry by the corresponding method. Higher geometry e. g. uses analytical, topological or algebraic methods. Excluding these methods one arrives at the so called “synthetical geometry” as elementary geometry. Again it is difficult to distinguish accordingly. E. g. length and area of elementary figures require the completeness of the real numbers which is a topological property.

It is remarkable that in mathematics “elementary” does not always mean “simple”. It is one of the advantages of higher mathematics that for important elementary theorems very simple proofs can be given by using methods of higher mathematics. In *trigonometry* (higher geometry) the *pythagorean theorem* (elementary geometry) is simply an inference of the *cosine theorem*. Among the numerous proofs of the *fundamental theorem of algebra* the elementary ones are the most difficult.

For Wittenberg “elementary” means *accessible* mathematical topics and methods in *school teaching*. Higher mathematics is taught at the university. And that does not mean that teaching at the *Gymnasium* must be old-fashioned. This is what Wittenberg wrote to Steiner in a letter:

„Was den Inhalt des Unterrichts anbelangt, so ist es mir vielmehr um eine tiefgreifendere Modernisierung zu tun, als sie auch Ihnen selber vorzuschweben scheint. Ich will mich also nicht mit der Anpassung an die formale Gestalt und die üblichen Methoden der modernen Mathematik begnügen, halte das tatsächlich nicht für übermäßig wichtig. Dagegen lasse ich mich aber sehr von zwei wichtigen modernen Gesichtspunkten leiten: das Eine ist gerade die kritische Analyse der modernen strukturellen Auffassung der Mathematik, und der Beziehungen zwischen Mathematik und Realität, mit der ich mich ja selber in meinem ersten Buch auseinandergesetzt habe. Das Andere sind die neueren Einsichten der Wissenschaftsphilosophen wie Polanyi und der Wissenschaftshistoriker wie Kuhn über das Wesen der Wissenschaft, die Rolle und die Gefahren des wissenschaftlichen Dogmatismus, usw. ...

So wie ich es sehe, geht es im wesentlichen um die Frage, ob wir den Schüler zu einer bedenkenlosen, besinnungslosen, letztlich dogmatischen Hinnahme der gegenwärtigen Denkweisen der Wissenschaft erziehen wollen, oder ob wir ihm im Gegenteil auch ein gewisses Maß der kritischen Distanzierung, des „Warum so und nicht anders“-Fragens mitgeben wollen.“ (Wittenberg to Steiner on January 29th, 1965)

(As far as the content of school teaching is concerned, a more profound modernisation seems to be more important to me than what you have in mind. Therefore, I don't want to content myself with the adaptation to the formal character and the usual methods of modern mathematics. Indeed, I don't regard this as being overly important. However, I am very much guided by two important modern aspects: The first is just the critical analysis of the modern, structural perception of mathematics and the relationship between mathematics and reality which I have considered myself in my first book. The other are the new insights of philosophers of science such

as Polanyi and the historians of science such as Kuhn, about the nature of science, the role and the danger of scientific dogmatism, etc. ...

As I see it, the question is essentially whether we want to educate the student towards an unscrupulous, senseless, ultimately dogmatic acquiescence to the present ways of thinking of science, or whether in contrast we also want to give him a certain amount of critical distance and questioning "why so and not in another way".)

Wittenberg was convinced that modern mathematics education does not need logic and set theory. Important elementary problems are only waiting for their discovery. As he wrote in *Bildung und Mathematik:*

> „Wesentliche Probleme des wissenschaftlichen Denkens liegen in der Schulstube vor uns und warten nur darauf, gesehen zu werden.“ (Wittenberg 1963[1], p. 235)
>
> *(Essential problems of scientific thinking are there in front of us in the classroom and are only waiting to be seen.)*

But in mathematics there is still another view of “elementary”. The fundamental concepts and properties of a mathematical theory are considered as the *elements* of this theory. As Proklus Diadochus (410–485), the last leader of Platon’s academy, wrote in his commentary on Euclid’s *Elements*:

> „Wenn wir aber das Ziel für den Studierenden feststellen, so werden wir sagen, daß eben das, was der Titel besagt, also eine grundlegende Einführung, ihm geboten sei und eine vollendete Ausbildung des Geistes der Studierenden für den gesamten Betrieb der Geometrie.“ (Proklus 1945, p. 215)
>
> *(But when we determine the objective for the student, we will say that it is exactly what the title states, that is, he will be offered a basic introduction and a complete formation of the student's mind for the entire operation of geometry.)*

He considered that book as a *fundamental introduction* to geometry. A fundamental introduction is possible for every mathematical theory. This is also true today. It deals with the fundamental concepts, relations, problems, and methods. And this was one aspect Steiner had in mind, when he wrote:

> „Das Elementare soll einerseits eine Schlüsselstellung für das Verstehen und Erfassen der charakteristischen Züge etwa eines Wissensgebietes einnehmen, also in prägnanter Weise das Prinzipielle des Gebietes widerspiegeln; es soll andererseits den Schüler für das jeweilige Gebiet aufschließen, sein Interesse wecken, es zu seiner Angelegenheit werden lassen.“ (Steiner 1969b, p. 49).
>
> *(What is elementary should, on the one hand, play a key role for understanding and grasping the characteristic features, for example, of a field of knowledge, and should therefore concisely reflect the principles of that field. On the other hand, it should open up the student for the particular field, arouse his interest, and let it become his own concern.)*

For Steiner "elementary" had two aspects: The elementary part of a mathematical field is the *basic part* of it. Therefore the structural concepts such as composition, group, ring, field, and isomorphism are elements of algebra.

And "elementary" does also mean *accessible for students*. It must be understood as a chance for teaching and consequently must be determined repeatedly. In this respect Steiner was influenced by the famous German educational scientist Wolfgang Klafki to whom he referred in his presentation at the annual MNU-Meeting in Aachen in 1964:

> „1. Das Elementare hat erschließenden Charakter in dem Doppelsinn, daß es dem Schüler Wirklichkeiten erschließt und ihn selbst zugleich für diese Wirklichkeiten aufschließt. ‚Das Elementare ist damit das doppelseitig Erschließende, in seiner erschließenden Funktion ist seine Fruchtbarkeit für die Bildung begründet.'
>
> *(What is elementary has a developing function in the dual sense that it offers realities to the student and at the same time opens him up for these realities. What is elementary has therefore a doubly developing role, its usefulness for education is justified in its developing function.)*
>
>> 2. ‚Das Elementare' muß ‚immer wieder neu bestimmt werden, und zwar eigenständig didaktisch im Blick auf die geistig-geschichtliche Wirklichkeit, in die der Schüler einst eintreten soll, und zugleich im Blick auf den Frage- und Verständnishorizont der jeweiligen Bildungsstufe.'

> *(What is elementary must be specified again and again, and in fact in a didactically self-contained manner in view of the intellectual-historically reality which the student will enter once, and at the same time in view of the horizon of questions and comprehension of the respective level of education.)*

3. ‚Das Elementare ist das am Besonderen zu gewinnende oder im Besonderen erscheinende Allgemeine.'

(What is elementary is what can be gained from the particular, or it is the general, appearing in the particular.)

4. ‚Das Elementare ist das' für die jeweilige Stufe ‚Prägnant-Einfache', das bei möglichst geringem Kompliziertheitsgrad gerade nicht das schlechthin Simple ist, sondern einen deutlichen Anspruch an das Verständnis eines allgemeinen Prinzips stellt, das eben am Elementaren besonders prägnant zur Anschauung gebracht werden kann." (Steiner 1964/65, p. 192)

(What is elementary is the concisely-simple for the respective level, that is, not what is simple as such at the lowest possible degree of complexity, but what makes a notable demand on the comprehension of a general principle that can be demonstrated, with what is elementary, in a particularly concise way.)

6. About the development of the reform

In the beginning the reform of mathematics education in Germany seemed to be successful despite many objections. The KMK-syllabus (KMK = Standing Conference of the Ministers of Education and Cultural Affairs of the Länder) of 1968 and as consequence the syllabi of the *Länder* chose the concepts of set, relation, mapping, function, and structure as *leading concepts*.

Numerous suggestions for teaching were made by didacticians, by devoted teachers, and textbook writers. To mention only a few examples the reader is referred to the contributions of Griesel, Kirsch, and Steiner for teaching the concept of group (Griesel 1965; Kirsch 1963, 1965; Steiner 1965c, 1966a). Textbook writers were stimulated by these ideas and developed new suggestions of their own. So the teaching of algebraic structures became an important topic of my book *Didaktik der Algebra* (Vollrath 1974).

But during the following years algebraic structures disappeared silently at first from teaching, then from the syllabi, and finally from the textbooks.

What were the reasons for the failure of the reform? In his paper *Zur Methodik des mathematisierenden Unterrichts* at a conference in Klagenfurt in 1976 Steiner criticized that teaching structures in schools mostly had been restricted to offer collections of examples of the corresponding structure. He judged, that this "classifying scholasticism" was misunderstood as "modern mathematics (Steiner 1976, pp. 213-214)."

It is my impression from discussions with teachers that they too saw these shortcomings. But they did not see any possibility to teach their students at school the *important* ideas of the underlying theories. Therefore most of the teachers regarded – and still regard – algebraic structures as essential to mathematics at the *university* level. Thus, they argued in the same manner as Wittenberg had done.

But also within university mathematicians the views have changed. Mathematics is not seen as a "science of the formal systems" any more as Steiner had taught (Steiner 1965b) referring to Haskell B. Curry (1900–1982). Even though mathematics is done on an abstract level of theory it is not an end in itself. Furthermore, the applications of mathematics have become more important. As a consequence, model building has taken up a central topic of mathematics. Therefore, it seems that mathematics developed in a direction that Laugwitz had expected.

The reform of mathematics teaching at the *Gymnasium* was just one aspect of the problem. A reform of mathematics instruction was necessary for the whole system of education. After the reform of mathematics teaching in the *Gymnasium* the New Math started as a general reform. It was a disadvantage that this reform of mathematics education imitated the new *Gymnasium* syllabus and used it for the whole secondary level (Damerow 1977). This can be read in the KMK-syllabus from 1968. The failure of New Math had also negative consequences for mathematics teaching at the *Gymnasium.*

But one must concede that the reform of mathematics education at the *Gymnasium* did not fail entirely. There are some worthwhile results which have become common usage, e.g. speaking of number sets, discussing properties of their operations, considering solution sets in solving equations, using functions for solving problems, studying geometric mappings and their composition, awareness for algorithms, and mathematization. It was mainly the influence of Steiner's friend Hans Freudenthal (1905–1990) that the more useful ideas of the reform for mathematics education were saved.

My book *Algebra in der Sekundarstufe* from 1994 tried to show a way of developing *algebraic thinking*, but algebraic structures did not play an important role any more. Hans-Georg Steiner was disappointed about my reactions on the developments in algebra teaching. He was convinced that the didactical potential of structures for mathematics education was not exhausted yet.

7. New perspectives

Through his international contacts and teaching experience abroad Steiner's view was not restricted to the *Gymnasium*. He developed new ideas for mathematics education, especially after he was chosen director of the "Zentrum für Didaktik der Mathematik" at Karlsruhe University in 1968.

He investigated the didactical problems of mathematics education on the primary level, when he was appointed as professor for didactics of mathematics in Bayreuth in 1970.

In 1975 he moved to the "Institut für Didaktik der Mathematik" (IDM) at Bielefeld University where he got the opportunity of a wider range of research on mathematical education. This was essential for him, because one of the main reasons for the failure of the reform had been deficits in didactical research (Steiner 1978).

His perspective of research can be seen in his paper about mathematizing (Steiner 1976) where he developed his key example of the voting- bodies from a much broader view of mathematics than in his early papers about that topic. But structures and axiomatics still appear in steps of the process of mathematization.

The reform of the upper secondary education in the FRG during the 70th offered Steiner and his working group the chance to develop a broad theoretical fundament for mathematics teaching that included research about innovation, professional work of the teachers, socialization, sociology of science, etc.

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