

67. On the appreciation of theorems by students and teachers

In: D.F. Robitaille, D.H. Wheeler, C. Kieran (Hrsg.) Selected Lectures from the 7th International Congress on Mathematical Education, Sainte-Foy (Université Laval) 1994, 353-363.

When a theorem has been taught, students are expected to understand it and to know a proof. They should be able to reproduce the theorem and its proof, and to apply the theorem correctly. But for a real understanding they need to know something about the historical background of the theorem about its place within the theory, and its relevance for applications. Therefore students should learn not only theorems but also the importance of these theorems. This can only be accomplished by teachers who have learned to appreciate theorems adequately. Therefore, an important part of teacher education must be concerned with the interpretation, discussion, and evaluation of theorems.

1. Discussing the Pythagorean theorem

At the beginning of my geometry lecture for future teachers I usually ask them which theorems they remember from their school geometry. Most years, the best-remembered theorem is the Pythagorean theorem. After I tell them that this is almost always the one selected by students, we try to find out why this theorem is so prominent. Typical comments by the students include the following:

This theorem is interesting (important, beautiful, highly regarded, surprising, central).

It has a simple (beautiful, impressive, suggestive, meaningful) formula.

The theorem concerns an important geometric figure, the right triangle.

These are very general judgments. In further discussion, more specific answers are given:

The Pythagorean theorem

- reveals a relationship between the sides of a right triangle.
- helps to express one side of a right triangle in terms of the other two sides.
- is a special case of the law of cosines.
- is an inference from the theorem: $a^2 = p \cdot c$; $b^2 = q \cdot c$.
- shows how to transform two squares into one square.

This theorem

- is named for Pythagoras, the Greek philosopher and mathematician
- was known to the ancient Egyptians.
- has been discovered in most cultures.

There are more than 200 proofs of the theorem, including one by GARFIELD, who became president of the United States.

To summarize, there are four general types of response:

- affective: beautiful, interesting, surprising;
- cognitive: special case, inference, reveals a relationship;
- instrumental: useful, applicable, helpful;
- cultural: known by Pythagoras, and the ancient Egyptians.

What are the origins of the students' appreciation for this theorem? We can presume that the most important source for their views is personal experience, gained by studying the theorem, its proofs and its applications. However, it seems likely that judgments by teachers have some influence as well.

But how can teachers teach adequate views of theorems? How effective are their methods?

2. Appreciation of theorems in mathematics instruction

It is helpful to understand how teachers can express their appreciation of a theorem to their students, either explicitly or implicitly.

Explicitly expressed appreciation of a theorem

It is traditional in mathematics to give hints about the importance of a proposition by identifying it as a lemma, corollary, theorem, or fundamental theorem. These assessments are handed down from generation to generation. They often have their origin in papers or books of the mathematicians who discovered the propositions. Well known examples include Gauss' *Theorema Egregium* or Sperner's Lemma. In the latter case, the lemma has become more famous than a normal theorem.

The teacher can give explicit expression to the appreciation of theorem by comments such as:

- This is an important theorem.

(which is a bland statement!) or by a more specific comment:

- This theorem is very useful for calculations concerning triangles.

In the second comment, assessment is directed to the use of the theorem whereas the following example expresses an appreciation for the knowledge gained by the theorem:

- The theorem expresses a relationship among the three sides of a right triangle.

An assessment about a theorem can also include a kind of reasoning about its importance:

- There are more than 200 proofs for the Pythagorean theorem. It is therefore one of the most prominent theorems in mathematics.

Sometimes the estimation of the importance of a theorem changes. A well known example is the "fundamental theorem of algebra" which is currently referred to as the "so-called fundamental theorem of algebra" in modern books on algebra. This makes clear that one should not overestimate these qualifications. But in both cases they express estimations explicitly.

In my personal experience, an explicitly expressed appreciation of a theorem is only impressive if it is specific, and based on reasons, knowledge, and experience.

Implicitly expressed appreciation of a theorem

In the name "Pythagorean theorem" special prominence is given to this theorem. The reference to a famous mathematician suggests that he discovered the theorem, though it is well known that this is often not true, as indeed it is not true for the Pythagorean theorem. Perhaps more importantly, the names of theorems can differ from country to country with a national identification. The name of a theorem can also refer to its contents, for example "mean value theorem", or "prime number theorem". In all these cases teachers can implicitly express their appreciation of the theorem.

But the way in which teachers deal with a theorem also reveals their appreciation of it. By starting with an interesting problem, discussing assumptions, giving different proofs, studying applications, or making remarks about its history, the teacher can bring the students to think: "This must be an important theorem because there is so much ado about it."

There is a strong conviction among mathematicians that the importance of a theorem is evident when it is really understood. Many mathematicians therefore avoid speaking or writing about their estimation of a theorem. For experts, their "hidden appreciation of a theorem" is recognizable in several ways. The position of a theorem within the theory, the numbers of references to a theorem, and the consequences drawn from a theorem all indicate appreciation.

Unfortunately many students feel lost when they are asked to express their

estimation of a theorem because they have not received clear hints that are relevant for judging it. Implicitly expressed appreciation of a theorem allows students a free hand to make their own judgments, but they must learn to interpret the teacher's behavior correctly.

Comparing explicit and implicit judgments can be summarized as follows. Explicit judgments of theorems are recognizable by the students. They reveal the personal preferences of teachers and ask for agreement, but can also invoke opposition. Above all, explicit judgments demand reasons. Implicit judgments allow students more freedom for their own assessment, but the students can also be misled by or misinterpret their teacher's behavior.

3. The problem of justification

A proposition is called a theorem if it is true relative to a system of axioms. The statement that a proposition is a theorem belongs to metalanguage, and can also be true or false. But what about the statement:

The theorem is important with respect to mathematical knowledge.

One may agree or disagree either on a rational or an emotional basis.

Some typical situations in which mathematicians are asked to evaluate theorems include theorems in a doctoral dissertation, theorems in a paper presented to a journal, theorems in a paper under review, comparing the "value" of a theorem in an award, or deciding which theorems shall be selected for a report in an encyclopedia.

There are not many statements by mathematicians about their standards. Let me give one example: BEHNKE (1966) wrote about the procedure for judging a research paper for a journal. Novelty and correctness of the results are necessary but not sufficient merits for publication. Criteria for the significance of a paper include:

- elegance of the presentation,
- ingenuity of the proofs,
- fertility of the considerations,
- adequacy of the resources,
- suitability of the reasoning.

But obviously each criterion is as vague as the quality which it is expected to judge. When BEHNKE characterized the qualified mathematician by the ability to apply these criteria correctly, the result was a circle between the judgment and qualification of a mathematician. After all, the community of mathematicians sets the values, and it is also responsible for the justification of the decisions. But the community pretends a harmony which is not always present.

Recent discussion about the status of the mean value theorem of calculus will illustrate the discord. VAN DER WAERDEN (1980) and LAUGWITZ (1990) judged the mean value theorem as:

- historically unimportant,
- clear by intuition,
- rather useful because of the conventions used in its proof,
- only interesting in its systematic aspects.

They concluded that the mean value theorem is rather unimportant.

SCHWEIGER (1987) and WINTER (1988), reviewing the same theorem, emphasized:

- it expresses practical intuitions from physics and economy,
- it opens a field of discoveries,
- it expresses the fundamental completeness of the real number system,

- it is important for approximations,
- it is a bridge from local to global changes,
- it is a paradox that the mean value theorem is equivalent to both a more special theorem (Rolle) and a more general theorem (Taylor),
- it is an example for a non-constructive theorem.

On this basis, they decided that the mean value theorem is very important.

Perhaps you will think that it is not so important whether the theorem is confirmed to be important as to know it. But from a didactic point of view this was a rather important discussion. The background was the question of the role the mean value theorem should play in a calculus course. The experts were mathematicians and didacticians who were influenced by their knowledge and experience, but also by their personal preference and taste. Their argumentation was impressive, though their emotions were rather irritating.

In my opinion there was not just one winner of this discussion. We all profited from it because we learned a lot about this theorem which we would not have found in textbooks. Perhaps teachers feel lost. What should they tell their students about the value of this theorem when the experts do not agree? But is it not an advantage to take part in an open discussion? It protects us from handling judgments of theorems dogmatically. The reasons given in arriving at the judgment help teachers in curriculum decisions, but they also reveal aspects for their own estimation of the theorem's importance.

4. Developing adequate estimations in student teachers

Mathematics books which are used at the university for the mathematics education of future teachers rarely comment on the assessment of theorems. While lectures are used to give more comments, in my experience students tend to relax during such commentaries. My remarks are often not seen as relevant for

the examination, even though an important task of courses in the didactics of mathematics is to discuss theorems which the students already know from their mathematics lectures, under the aspect of evaluation. Again, I demand that the students get the chance to reflect on their experiences, to listen to other students' judgments, and to consider them carefully. Usually it is very surprising for the students to realize that people can have different opinions about mathematical facts!

I would like to invite mathematicians, when they are writing books for future teachers, to comment more about theorems from different points of view, and on specific ways of reasoning. My request of the didacticians is that they discuss questions of evaluation in an open way without being dogmatic.

As we have seen, the appreciation of a theorem refers to four aspects: knowledge, usage, culture, and beauty. It is rather easy for the students to judge the efficiency of a theorem because they have only to remember their own use of the theorem. Therefore it is not surprising that the assessments of student teachers are mainly directed to usage.

Students are able to discover the knowledge provided by a theorem by reflecting on it for a while. It is well known that consideration of the problem context which led to the discovery of a theorem enables student teachers to appreciate the theorem in the context of a culture. But the realization often seems to be not worth the effort for the students.

Questions about the beauty of a theorem are sometimes irritating to student teachers, though my students were very interested in David Wells' (1988, 1990) investigation about the evaluation of theorems by the readers of *The Mathematical Intelligencer*. Each of 24 prominent theorems had to be given a score for beauty. The winner was Euler's identity. Teachers should be aware that there are many books and papers about the beauty of mathematics which can stimulate students and teachers.

In summary, student teachers need explicit comments and discussions about the

aspects of knowledge, usage, beauty, and culture to develop adequate estimations of theorems.

5. Appreciation of theorems by students

When theorems are taught at the gymnasium, teachers are used to discussing them. We were interested in the student assessments of theorems that resulted from this. We interviewed students from Grade 8 and Grade 10 about their estimations of Geometry theorems, and students from Grade 13 about calculus theorems. For the 8th graders Thales' theorem – The angle in a semicircle is a right angle – and the congruence theorems for triangles were the most prominent. Thales' theorem was interesting to them because of its use in constructions. The congruence theorems were seen as important for proofs, and as a basis for the construction of triangles.

The 10th graders appreciated the Pythagorean theorem and Thales' theorem most. They reasoned that they are logical, easy to understand, often used in tests, and used in constructions. The appreciation of the Pythagorean theorem resulted from tests, the great numbers of problems solved in connection with this theorem, the great variety of examples, and the impressive formula.

In our interviews with the 13th graders we asked them and their teacher about their appreciation of the calculus theorems. The most important theorems for these students were the theorems about minima and maxima, L'Hôpital's rule, and the theorems of limits.

The most important theorem for the teacher was the fundamental theorem of calculus. The differences of the assessments between students and teacher resulted from their different viewpoints. The students' interest was more directed to usage while the teacher's interest was more directed to knowledge.

Such differences can appear quite dramatically. I remember a classroom situation from my own teaching in Grade 7. When I became very enthusiastic about

a theorem, a girl jumped up and cried: "This is all rubbish!" This was an evaluation too!

Our appreciation of theorems may provoke our students to protest. And their rejection can be a provocation for the teacher. How should we react adequately? I think we can agree that it is useless trying to convince the students about the importance of a theorem. Why not let the students know that they are allowed to have different views? Perhaps they will discover the importance of the theorem by themselves. On the other hand, it is also true that many students like mathematics because of its objectivity. They get the chance to appreciate mathematics based on their own criteria and decisions.

In summary, in working with an important theorem, teachers should try to balance the different aspects of knowledge, usage, beauty, culture. They should become aware of the students' appreciations and should accept them as expressions of their personality. But they also should give their students a chance to make adequate estimations of theorems by reasoning without being dogmatic or autocratic.

6. Balanced teaching

When I recently asked my students about their appreciation of theorems from their school mathematics, one student said (and many agreed): "Mathematics instruction was not theorems. It had more to do with techniques." They therefore felt rather lost at my question about their appreciation of theorems. It seems to be more important for students, and perhaps for their teachers too, that a method works, rather than to know why the method works. It is more comfortable, and with respect to tests and examinations more effective. But the result is unbalanced teaching.

We emphasized different aspects of significance. Obviously these aspects have to be balanced in mathematics education. There must be a balance between knowledge and usage, theory and practice, beauty and rigor, culture and techni-

que. One-sided assessments can reveal unbalanced teaching. But it is also true that balanced estimations of theorems can help to balance different aspects of teaching. They can help the students to gain a valid impression of mathematics. Thus, balanced assessments play a key role in teaching.

But is it not a question of the subject matter? In a geometry course, there are many theorems which express knowledge and a few concerning techniques. But in an algebra course in secondary schools there are usually "laws", and "formulas" and, above all, techniques for transforming expressions and solving equations; but only a very few "theorems", such as the binomial theorem, or Vieta's theorem (about the relationship between the roots and the coefficients of a quadratic equation).

There are different traditions of teaching geometry and teaching algebra with respect to theory. This is also true for the history of mathematics. Axiomatic presentations of arithmetic and algebra appeared rather late. HILBERT's *Foundations of geometry* and LANDAU's *Foundations of calculus* can be seen as the culmination of this development, offering equivalent presentations in geometry and arithmetic. From LANDAU's book one learns that a large number of propositions in arithmetic can be treated as theorems, which is not common in mathematics instruction. To better balance geometry and algebra teaching I suggest writing, for example, the law of commutativity of multiplication, or the rule for adding fractions, or the formula for the solution of a quadratic equation as theorems.

Above all, properties which are fundamental for the understanding of arithmetic and algebra should be pointed out as theorems. As illustrations, consider:

- Natural numbers can be presented as sums of units.
- Real numbers can be presented as the limits of sequences of rational numbers.
- The square of a real number cannot be negative.

Students can only develop a valid impression of mathematics if they receive a balanced teaching in which they can appreciate theorems as a distillate of knowledge and potential.

7. Accentuated teaching

We started with an outstanding theorem. But every theorem can be appreciated with respect to cognition, usage, culture, and appearance. To some extent each theorem is important. If a certain theorem were omitted in an axiomatic theory it could be critical for the whole theory. However, if teachers call every proposition an important theorem, this is not credible. It would have the same effect as underlining every word in a book (as some readers appear to do). "If everything is important, then nothing is important." (SHENITZER 1986).

Nevertheless, to illustrate properties by appropriate theorems helps students in several ways. They become aware of what is noteworthy, find out what they are expected to know, and develop a basis to which they can refer when they are trying to prove a statement. However, it is also necessary to differentiate between theorems so that students can recognize the structure of a subject area, become aware of the key properties, and develop standards.

To give special prominence to a theorem, say by referencing it to a mathematician, helps students appreciate the achievement of mathematicians and understand their contributions to culture. Emphasizing importance of theorems may help students to appreciate that mathematics is something important for culture and for themselves.

As a consequence, students need a kind of teaching in which they get a chance to distinguish between important and less important facts. They can only develop standards when they become acquainted with the really outstanding results of mathematics.

8. Steps towards adequate estimations of theorems

We understand the appreciation of a theorem as a part of the meta-knowledge that we want students to develop in mathematics education. Students learn to reflect upon theorems by asking questions such as:

- What does the theorem represent?
- What is the essential point of the theorem?
- What consequences does this theorem have?
- What problems can be solved with this theorem?

Students can initiate their assessments of a theorem by tasks such as:

- Trying to formulate the theorem in your own words.
- Giving a descriptive title for the theorem.
- Trying to find a suitable name for the theorem.

Mathematical knowledge is often tested through problem solving. For testing students' meta-knowledge it seems to be more convenient to let the students write an essay about the theorem. This is not very common in mathematics instruction. Writing mathematical essays was recommended in Germany by M. Wagenschein, but students are rarely asked to do so. Problem solving is still predominant in German schools.

Finally, I think it is very important that students have a chance to discuss their assessments of theorems with other students and with their teacher. They should be willing to listen to other students' reasons, to give reasons for their appreciation of a theorem, and be prepared perhaps to change a personal assessment during discussion.

Discussing assessments of theorems is a training method and a test for scientific culture. It can be seen as a contribution to "mathematical enculturation" (BISHOP 1988).

Acknowledgments

I wish to acknowledge with gratitude the influence of ALEXANDER ISRAEL WITTENBERG on my philosophy of education. He was professor at Laval University in Québec, and later on at York University in Toronto. His book, *Bildung und Mathematik* (1963), is a program for mathematics education through a genuine experience of mathematics. I fear most didactician do not know what they are missing by not having an edition in English. I wish to close with a statement that accords with Wittenberg's philosophy:

Students and student teachers have the right to learn in what respect the theorems they are expected to learn are important.

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