51. The role of mathematical background theories in mathematics education

In: H.-G. Steiner, A. Vermandel (Hrsg.), Foundations and Methodology of the Discipline Mathematics Education (Didactics of Mathematics), Proc. 2nd TME-Conference, Bielefeld, Antwerpen 1988, 120-137.

1. The concept of background theory

A theory of mathematics education should provide a rational decision making process for the development of a curriculum for choosing and arranging the content, offering necessary simplifications and posing possible applications. A rational basis for making decisions like these has been a mathematical "background theory" in the didactic of mathematics. This concept was introduced and has often been used by German didacticians. Let us begin by looking at two typical examples.

KIRSCH (1972) offered a "didactically oriented system of axioms for elementary geometry". His approach is set forth as follows (p.139):

> "In the following we present an approach to elementary geometry which is mathematically correct and is intended primarily to be an aid for planning geometry teaching from age 11-14. It is thought of as an orientation for teachers; they should have the axiomatic system 'under the table' without discussing each particular point with the students. They ought to be able to translate the axiomatic theory easily into the language of mathematics teaching."

An axiomatically-founded geometry course for teachers was written by HOLLAND (1974) which was intended as a background theory for geometry teaching from age 11-16. He described the role of a background theory as follows (p.9):

> "Obviously, a deductive approach to geometry which is thought to be a background theory for geometry teaching has to take into consideration a

> didactical point of view when choosing the axioms, arranging them, and for building up the whole theory. This didactical point of view results from investigations about the learning processes of students.”

Let us look at the concept "mathematical background theory". The reference to "mathematical theory" implies a need for a mathematical foundation of mathematics instruction. Hence, there is a close connection between mathematics and instruction. The word "background" implies some necessary "distance" between mathematics and teaching due to educational considerations. Didacticians who refer to background theories are implicitly convinced that a perceived gap between mathematical requirements and pedagogical practice can be narrowed by use of a suitable mathematical background theory. Thus identifying or developing a convenient background theory is a matter of research in mathematics education. Comprehension of this concept has developed in the didactics of mathematics as has the involvement of didacticians in developing background theories for mathematics education. By use of an historical analysis below we try to identify phases in the development of this concept.

2. Background theories for geometry teaching

For many centuries EUCLID's Elements served as the background theory for teaching geometry. Curriculum decisions have been based on this theory as the main authoritative source. Let us consider, for example, a very interesting textbook by J.A.C. MICHELSEN (1781): *Versuch in socratischen Gesprächen über die wichtigsten Gegenstände der ebenen Geometrie.* The book attempts to offer geometry to students in the manner of socratic dialogues. As the author remarked, the book was the result of real dialogues between him as the teacher and his students. This is a very origina1 textbook from a methodo1ogica1 point of view. MICHELSEN was a reputable professor of mathematics and physics at the Gymnasium in Berlin. But the book humbly followed EUCLID's Elements as can be seen in the table below which MICHELSEN took as a justification for his approach (pp.171-174).The left column refers to EUCLID's Elements Book I, and the numbers refer to definitions (Erklärungen), postulates (Forderungen), axioms (Grundsätze), theorems (Lehrsätze), and problems (Aufgaben). The right co1umn consists of the related parts of the book (Versuche...). There are just a few divergences in the beginning which are indicated in the table.

Verzeichniß

der

Erklärungen, Forderungen, Grundsätze, Aufgaben und Lehrsätze in den vier ersten Büchern der Elemente des Euclides, welche in diesen Versuchen vorkommen.

Die römischen Zahlen unter dem Titel: Euclids Elemente, zeigen das Buch, und die teutschen die Erklärungen, Forderungen u. s. w. an. Die Zahlen unter dem Titel: Versuche u. s. w. haben dieselbe Bedeutung, die sie in dem vorhergehenden Verzeichniß gehabt haben.

Euclids Elemente			Versuche u. s. w.
Erklärungen.			
I.	1.	—	—
—	2.	—	I. 8 - 10.
—	3.	—	—
—	4.	—	I. 11 - 14.
—	5.	—	I. 19 - 21.
—	6.	—	—
—	7.	—	I. 21 - 22.
—	8.	—	I. 30. 31.
—	9.	—	I. 29. 30.
—	10.	—	I. 33 - 35.

Zu

Euclids Elemente			Versuche u. s. w.
I.	11.	—	I. 31 - 35.
—	12.	—	I. 33 - 35.
—	13.	—	—
—	14.	—	I. 39 f.
—	15.	—	I. 48. 49. III. 86. 87.
—	16.	—	— —
—	17.	—	III. 87. 88.
—	18.	—	III. 91.
—	19.	—	III. 91.
—	20.	—	I. 42. 43.
—	21.	—	I. 44.
—	22.	—	I. 44.
—	23.	—	I. 44.
—	24.	—	I. 44. 45.
—	25.	—	I. 44. 45.
—	26.	—	I. 44. 45.
—	27.	—	I. 48.
—	28.	—	I. 48.
—	29.	—	I. 48.
—	30.	—	II. 117. 118.
—	31.	—	II. 117. 118.
—	32.	—	II. 117. 118.
—	33.	—	II. 117. 118.
—	34.	—	—
—	35.	—	II. 98. 99.
Forderungen.			
I.	1.	—	I. 56, 68.
—	2.	—	I. 56. 68.
—	3.	—	I. 56. 57. 68.

Zu

Euclids Elemente			Versuche u. s. w.
Grundsätze.			
I.	1.	—	I. 63. 68.
—	2.	—	I. 63. 64. 69.
—	3.	—	I. 64. 65. 69.
—	4.	—	—
—	5.	—	—
—	6.	—	—
—	7.	—	—
—	8.	—	I. 66. 70.
—	9.	—	—
—	10.	—	—
—	11.	—	I. 85. f.
—	12.	—	—
Aufgaben und Lehrsätze.			
I.	1.	—	I. 60. 61.
—	2.	—	—
—	3.	—	I. 57. 58.
—	4.	—	I. 71. 76.
—	5.	—	I. 77. 88 - 96.
—	6.	—	I. 101 f.
—	7.	—	I. } 108.
—	8.	—	I. } 108.
—	9.	—	II. 4. 21 f.
—	10.	—	II. 4. 31 f.
—	11.	—	II. 4. 33.
—	12.	—	II. 4. 34 f.
—	13.	—	II. 23 f.
—	14.	—	II. 28 f.
—	15.	—	II. 26 f.

Zu

Euclids Elemente			Versuche u. s. w.
I.	16.	—	II. 38.
—	17.	—	II. 48 f.
—	18.	—	II. 54 f.
—	19.	—	II. 58 f.
—	20.	—	II. 60 f.
—	21.	—	II. 65 f.
—	22.	—	II. 62.
—	23.	—	II. 63.
—	24.	—	II. 72 f.
—	25.	—	II. 77 f.
—	26.	—	II. 87 f.
—	27.	—	II. 99 f.
—	28.	—	II. 99 f.
—	29.	—	II. 105 f.
—	30.	—	II. 110.
—	31.	—	II. 111 f.
—	32.	—	II. 112. 113.
—	33.	—	II. 114.
—	34.	—	II. 115. 126 f.
—	35.	—	II. 128 f.
—	36.	—	II. 131 f.
—	37.	—	II. 133 f.
—	38.	—	II. 133 f.
—	39.	—	II. 137 f.
—	40.	—	II. 137 f.
—	41.	—	II. 140 f.
—	42.	—	II. 142.
—	43.	—	II. 142 f.
—	44.	—	II. 147.

Zu

Thus, we identify

Phase 1: EUCLID's Elements as the sole background theory for geometry teaching.

Subsequently a new geometric approach was given by MÖBIUS (1827) in his book *Der baryzentrische Kalkül*. Transformation geometry became the "modern" geometry and textbook writers started to regard this geometry as a background theory. An early example is C.A. BRETSCHNEIDER's book (1844). The organization of his *Lehrgebäude der niederen Geometrie* is rather close to the ideas of MÖBIUS.

1 Synthetische Geometrie

a) Geometrie der Lage (site)

b) Geometrie der Gestalt (shape)

c) Geometrie des Maßes (measurement)

2 Analytische Geometrie

a) Goniometrie

b) Trigonometrie

c) Koordinatengeometrie

This textbook was far removed from the needs and abilities of students. The mathematics was prominently dominating. But this was not necessarily the only way to teach transformation geometry. For FELIX KLEIN (1908) the transformations were very close to actions.

He therefore saw the possibility that geometry could be taught by motions, following the psychological principle of adapting instruction to the children's development. Under KLEIN's influence transformation geometry became very important for mathematics education. We regard this as

Phase 2: Transformation geometry as the modern background theory for geometry teaching.

At the end of the last century HILBERT's Grundlagen der Geometrie (1899) had brought to perfection EUCLID's foundation of geometry. But his foundation of geometry stimulated many mathematicians to develop new foundations. The situation was substantially changed in that different mathematical theories as possible background theories came into existence. Didacticians now had to choose a special theory as background theory from among several competing theories, moreover, they had to provide rational justification for their decisions. A typical question was: Which axioms are most convenient for teaching geometry in the secondary school? To illustrate this, we cite some examples from a paper by WILLERS (1922).

> "All axiomatic systems based on the opinion that 'the space is a number-manifold in which each point is given by three coordinates and vice versa', are useless for school education" (p.69).

> "Each axiomatic theory of vectors, as introduced by GRASSMANN and whose basic concepts and theorems were developed by PEANO is insufficient for mathematics education as well" (p.70).

> "The system of PIERI is not applicable for schools. Since PIERI does not use the concept of ordering, the postulates are difficult and complicated" (p.71).

> "The rigid systems of HILBERT, VEBLEN, PEANO (Prinzipii), R. MOORE, and SCHWEIZER, and the ones of VERONESE (Elementi), INGRAMI, and RAUSENBERGER (containing many mistakes) which were made especially for mathematics teaching are not suitable with regard to the concept of congruence for modern school mathematics" (p.75).

> "In the following paragraph a system is developed which appears didacti-

> cally practical. It uses point, line segment, and reflection as fundamental concepts and introduces motion as a derived concept" (p.77).

WILLERS used didactical arguments against different axiomatic approaches. But we can see there is no doubt about the role of an axiomatic theory as background theory (p.68):

> "Geometry teaching can rely on visual perception in the beginning. However, it can not do without a correct mathematical basis for the teacher."

Now we have

Phase 3: Different axiomatic theories as competing background theories for geometry teaching

But there was a rather large gap between geometry instruction and the background theory due to the cognitive limitations of students. It would seem that textbook writers could only follow such a background theory "from a distance", so to speak.

The didactics of mathematics began to take form during the 1960's and people involved in mathematics education simultaneously developed more confidence. They now started to develop new axiomatic theories convenient for *teaching purposes* in order to minimize the perceived discrepancy between background theory and instruction. During this period didacticians had the task both of developing axiomatic theories and of reasoning for their utility as background theories for instruction. This development opened an interesting approach for research in the didactics of mathematics for mathematicians who were involved in mathematics education This kind of research made it easier for didacticians to communicate with mathematicians and to also establish didactics of mathematics as a research discipline.

Consider again, for example, HOLLAND's (1974) ideas about his axiomatic approach to elementary geometry (p.7):

> "This book presents for the first time a complete axiomatic approach of Euclidean plane geometry. Its system of concepts and its choice and organization of the geometrical content is closely oriented to recent geometry teaching in school as it is presented through textbooks curricula, and catalogues of objectives of each county."

This statement is typical and helps us to identify

Phase 4: An axiomatic theory derived by didacticians from practice of teaching as background theory for teaching geometry.

From these axiomatical approaches came linearizations of curricula and restriction to relevant concepts, problems, and method, which, in turn, restricted a student's view and understanding of geometry. In relation to teaching practice, it had the tendency to isolate the school world The wide variety in the world of geometry did not appear and seemed to have no chance to emerge. Therefore this development led to a crisis during the 1970's.

The weaknesses of this approach were highlighted by FREUDENTHAL (1973) in a discussion about the angle concept:

> "As has been stressed several times, there is more than one angle concept. Some didacticians claim that there is only one which is correct. Love of order is fine unless it goes as far as to forbid important concepts because they do not fit into the system. Properly said such would be a bad mathematical attitude. It has cost a great deal of trouble to get mathematicians used to the fact that there are various number concepts, which are now carefully distinguished from each other. If rather than being distinguished all angle concepts but one are forbidden, pupils will never learn to distinguish them – forbidding rules never work (476).

> "At least three angle concepts are practically, and thus didactically, important. Systematizing mathematicians are prone to restrict themselves to one and to eliminate the others. It is the same mentality that led the Greeks to restrict themselves to integers and to ban fractions. Of course, in every day

> life and in numerical mathematics one used fractions though in pure mathematics they were forbidden. It is the same schizophrenic attitude as that of a mathematician who recognizes the goniometric angle concept only, but of necessity speaks of the visual angle under which he sees an object and who knows very well that a half turn and a turn and a half is not the same when he turns a key in a keyhole. I admit there are people who are not convinced by such arguments. To their view the existence of instruments to measure angles is rather an argument against angles in mathematical instruction. Their suspicion is in particular aroused by the cyclic orientation of the half-lines pencil and that angle concept I termed analytic."(p. 487).

It is clear from these statements that one special axiomatic theory is not sufficient as background theory. For the teaching of geometry, a possible way out of this difficulty might be an extension of the concept of background theory. Perhaps we can take background theory as the total variety of relevant concepts, possible sets of axioms theorems, problems and applications of a mathematical area (VOLLRATH, 1979). For instance "group theory" is not restricted to a special axiomatic representation of the theory, but includes all knowledge about groups. Under this general aspect the background theory can assist us in answering questions such as: Which are the most important concepts of the theory? Which are their most important properties? Which methods are typical for this theory? Which theorems are important for understanding the relationship between the main concepts? Which theorems are the basis for efficient algorithms? Which problems led to the historical development of the theory? Which are the most important applications of this theory? It can also help to decide whether a given definition, theorem or proof is correct, and whether d Simplification is adequate. The answers Can help in making curricular decisions and in choosing, arranging and representing content of the theory for instruction. This interpretation of a background theory may help to establish a great variety of problems, methods, and ideas in mathematics instruction. Further, it may help to overcome the limitations of a special axiomatic approach. This leads us to identify

Phase 5: The totality of geometric knowledge, including the ideas connections, applications and evaluations, may provide the background theory for teaching geometry.

The mathematician working in geometry is most often a specialist whose main interest lies in a special field of geometry. Certainly it is an important task for the didactician to use the knowledge of the specialist and to get an overview of the whole area. But the view of the didactician cannot be solely directed towards mathematicians. There are many different aspects of our cultural world under which geometry is of some interest: e.g., geometry as a inventory of axiomatic theories; geometry as a reservoir of strategies for solving problems; geometry as a theory of real space; geometry as a theory for actions; geometry as a result of cultural history; geometry as investigation of forms (VOLLRATH, 1976).

Mathematics education should take into account more than just the axiomatic aspect when a curriculum is planned. Indeed, it is the scholarly and professional responsibility of the mathematics educator to exercise precisely this function.

3. Back ground theories for teaching fractions

EUCLID worked with proportions and avoided fractions. It was a long time before fractions were regarded as numbers. Parallel to the foundations of geometry, the foundations for numbers were laid at the end of the last century. It is difficult to discern whether it was a disadvantage for children that no Greek tradition existed for teaching fractions. But as a result, the teaching of fractions did not evolve as dogmatically as geometry teaching. But even though, a tradition developed. Two operations led to fractions: dividing and measuring. This occurred as a consequence of daily life; people dealing with money, length, area, volume, weight, and time. On a more advanced level fractions were regarded as numbers.

Some inspiration came from pedagogues who integrated the teaching of fractions into their pedagogical systems,e.g., PESTALOZZI (1746-1827). More contributions were made by didacticians at the beginning of the present century when they developed theories of mathematics education and applied these theories to teaching fractions e.g., KÜHNEL (1916). But these development gave rise to difficulties from their interpretations: "two thirds of" had to be translated into "times two thirds". Hence multiplication and division were infused into the process along with the difficulties associated with teaching these concepts. Much effort was expended to overcome these difficulties. It is significant that the solution was not sought through mathematics, but rather, through *methodology*.

A typical introduction to fractions can be seen in the illustration below from a textbook (FLADT, KRAFT, DREETZ, 1959, p.ll).

The teaching of fractions became an undisputed part of arithmetic in which fractions were defined as quotients of natural numbers. Similarly, addition, subtraction, multiplication, and division became an extension of operations on the natural numbers to the positive rational numbers This was often justified by a reference to HANKEL's "permanence principle". Thus we have

Phase 1: The theory of fractions as part of the traditional arithmetic as the sole mathematical background theory for content; but with the didactics of mathematics being responsible for the method of teaching.

This phase corresponds to the first phase of geometry teaching, i.e., with just one background theory. But its influence on curricular decisions was not nearly as strong as in geometry. From this came a rather naive treatment of fractions guided by practice. A number of didactical inventions were made to diminish the difficulties.

During the 1960's didacticians felt the need to more firmly construct fractions as part of the foundations of the number system in mathematics (FREUND

§ 3. Entstehung und Veranschaulichung der Brüche. 11

Ein Vieleck heißt regelmäßig, wenn es lauter gleiche Seiten und gleiche Winkel hat.

Durch die Ecken eines regelmäßigen Vielecks kann man einen Kreis legen. Er heißt „Umkreis" des Vielecks.

II. Das Rechnen mit gemeinen Brüchen.

§ 3. Entstehung und Veranschaulichung der Brüche.

Der Bruch als Teil eines Ganzen.

1. Ein Bäcker teilt eine Torte in 2, 4, 8, 16 gleiche Teile. So entstehen 2 Halbe, 4 Viertel, 8 Achtel, 16 Sechzehntel der Torte.

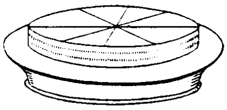


Bild 23.

2. Zeichne Kreise mit dem Halbmesser 3 cm und teile sie nach Augenmaß in a) 2 b) 3 c) 4 d) 5 e) 6 f) 7 gleiche Teile. Prüfe die Teilung mit dem Zirkel oder mit dem Winkelmesser nach.

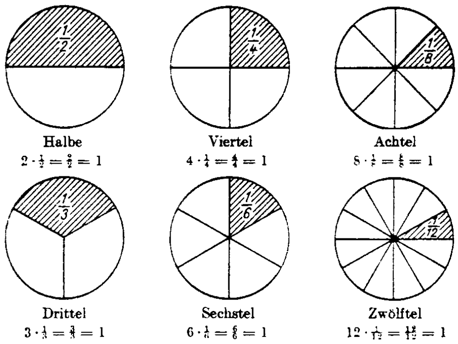


Bild 24.

3. Zeichne eine Schokoladentafel (Bild 25) und gib an, wie du sie — möglichst auf verschiedene Weise — in a) Halbe b) Drittel c) Viertel d) Sechstel e) Zwölftel f) Vierundzwanzigstel brechen kannst.

1965/66). Accordingly different axiomatic characterizations were developed for positive rational numbers as well as methods for constructing them through natural numbers. An axiomatic theory of the positive rationals based on the axioms of an ordered semifield was rather close to the traditional background theory.

On the other hand, fractions had to be introduced as an extension of the natural numbers. This approach had its origin in a construction through natural numbers. Didacticians in the German Democratic Republic tried to make this approach elementary (TIETZ et alii, 1969, p.29) as can be seen below.

Alle Brüche, die durch Kürzen oder Erweitern auseinander hervorgehen, fassen wir zu einer Klasse zusammen.

DEFINITION: Alle Brüche, die durch Kürzen oder Erweitern auseinander hervorgehen, bilden eine Klasse. Jede solche Klasse heißt *gebrochene Zahl*.

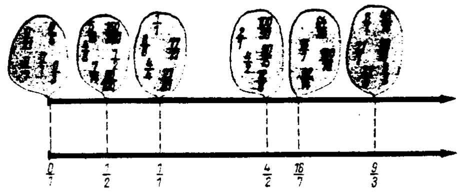
Wenn wir z. B. von der „gebrochenen Zahl $\frac{1}{2}$" sprechen, so meinen wir also diejenige Klasse von Brüchen, in der u. a. der Bruch $\frac{1}{2}$, aber auch die Brüche $\frac{2}{4}$, $\frac{3}{6}$ usw. liegen.

Sprechen wir von dem „Bruch $\frac{1}{2}$", so meinen wir nur das aus den natürlichen Zahlen 1 und 2 gebildete geordnete Paar.

Die verschiedenen Brüche $\frac{1}{2}$ und $\frac{2}{4}$ geben dieselbe gebrochene Zahl an, also dieselbe Klasse von Brüchen. Wir können deshalb schreiben: $\frac{1}{2} = \frac{2}{4}$

Häufig wird zur Angabe einer gebrochenen Zahl derjenige Bruch benutzt, dessen Zähler und Nenner teilerfremd sind.

Das Kürzen oder Erweitern eines Bruches bedeutet, daß man von diesem Bruch zu einem anderen Bruch aus derselben Klasse übergeht, d. h. dieselbe gebrochene Zahl durch einen anderen Bruch angibt (Bild B 2).



B 2

Bezeichnen wir die Menge der gebrochenen Zahlen etwa mit R^*, so gilt z. B.:

$$\frac{1}{3} \in R^*; \quad \frac{0}{7} \in R^*.$$

Um gebrochene Zahlen an einem Zahlenstrahl darzustellen, gehen wir folgendermaßen vor:

Wir tragen vom Anfangspunkt eines Strahls aus eine Strecke mit beliebiger Länge ab (Einheitsstrecke). An den Anfangs- bzw. Endpunkt dieser Strecke schreiben wir die Brüche $\frac{0}{1}$ bzw. $\frac{1}{1}$ zur Bezeichnung der entsprechenden gebrochenen Zahlen. Von dem Endpunkt der Einheitsstrecke ausgehend, tragen wir Strecken mit derselben Länge fortlaufend ab und schreiben an die erhaltenen Punkte der Reihe nach die Brüche $\frac{2}{1}$, $\frac{3}{1}$, $\frac{4}{1}$, usw. (Bild B 3). Durch Abtragen von Bruchteilen der Einheitsstrecke finden wir entsprechend die Punkte, die beliebigen anderen gebrochenen Zahlen zugeordnet sind. Dadurch wird jeder gebrochenen Zahl ein Punkt des Strahls zugeordnet.

Didacticians were now faced with a choice between an axiomatic and a constructive approach. This problem was part of an ardent discussion about the role of axiomatics in mathematics education in 1965 (LAUGWITZ, 1965, STEINER, 1965). No clear-cut resolution emerged from this dispute.

Thus we identify

Phase 2: Different approaches to fractions –constructive or axiomatic – as competing background theories for teaching fractions.

There was significant agreement for the need of more mathematical rigor in both method and formulation. It was very important, for example, to distinguish between "fraction" and "fractional number", (which led to inexpressible rules). All these attempts were rather far from tradition and at the same time, rather inadequate for teaching. Therefore didacticians sought a mathematical approach which was closer to tradition and which would help to overcome the well-known difficulties with multiplication and division. A solution was found by the discovery of the "operator" concept at the end of the 1960's. A relevant background theory was elaborated by such didacticians as GRIESEL (1968), KIRSCH (1970), PICKERT (1968) which transformed an idea of H. WEYL (1918).

This new background theory could easily be made more elementary by means of a "machine" model . Helping to stimulate this approach was the "stretchers and shrinkers" idea of BRAUNFELD (1968). The influence of this approach can be seen in a textbook below (GRIESEL, SPROCKHOFF, 1974 p.26). We now have

Phase 3: The operator approach developed by didacticians as suitable background theory for teaching fractions.

The resulting curricula and the relating textbooks however suffer from an expanding net of artificial concepts and complicated calculations with operator chains which, at the same time, neglected certain aspects of fractions. For example multiplication was introduced before addition, which was rather unnatural for most teachers and students.

Bruchoperatoren

1. Fülle die Tabellen aus und gib, wenn möglich, die Ersatzmaschine an.

a) $\boxed{:3}$ — $\boxed{\cdot 9}$

15 cm		
6 dm		
27 cm		

b) $\boxed{\cdot 2}$ — $\boxed{:3}$

15 cm		
6 dm		
27 m		

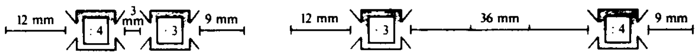
c) $\boxed{:3}$ — $\boxed{\cdot 2}$

	5 cm	
	2 dm	
	9 m	

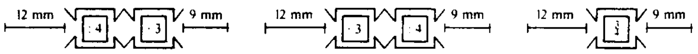
d) $\boxed{\cdot 4}$ — $\boxed{:6}$

	60 cm	
	24 dm	
	108 m	

Eine $\boxed{:4}$-Maschine und eine $\boxed{\cdot 3}$-Maschine sind miteinander gekoppelt. Diese Anlage kann weder durch eine Mal-Maschine noch durch eine Durch-Maschine ersetzt werden.



Wir erfinden eine neue einzelne Maschine für eine Anlage aus einer $\boxed{:4}$- und einer $\boxed{\cdot 3}$-Maschine



Das Programm der neuen Maschine bezeichnen wir kurz auch mit $\boxed{\frac{3}{4}}$ (gelesen: dreiviertel).
Die neue Maschine heißt **Bruchmaschine**. Das Zeichen $\frac{3}{4}$ nennt man einen **Bruch**.

Ihr Programm $\boxed{\frac{3}{4}}$ heißt **Bruchoperator**.

3 → **Zähler** des Bruches
— → **Bruchstrich**
4 → **Nenner** des Bruches

2. Fülle die Tabellen aus und vergleiche.

a) $\boxed{:5}$ — $\boxed{\cdot 3}$

45 m		
105 m		
165 m		
225 m		

$\boxed{\frac{3}{5}}$

45 m	
105 m	
165 m	
225 m	

$\boxed{\cdot 3}$ — $\boxed{:5}$

45 m		
105 m		
165 m		
225 m		

b) $\boxed{\cdot 5}$ — $\boxed{:3}$

45 m		
105 m		
165 m		
225 m		

$\boxed{\frac{5}{3}}$

45 m	
105 m	
165 m	
225 m	

Der Mal-Operator bestimmt den Zähler des Bruches,
der Durch-Operator bestimmt den Nenner des Bruches.

○ $\boxed{:5} \circ \boxed{\cdot 3} = \boxed{\cdot 3} \circ \boxed{:5} = \boxed{\frac{3}{5}}$; $\boxed{\cdot 5} \circ \boxed{:3} = \boxed{:3} \circ \boxed{\cdot 5} = \boxed{\frac{5}{3}}$

$\boxed{:5}$ —— $\boxed{\cdot 3}$ $\boxed{\cdot 5}$ —— $\boxed{:3}$

$\boxed{\cdot 3}$ —— $\boxed{:5}$ $\boxed{:3}$ —— $\boxed{\cdot 5}$

$\boxed{\frac{3}{5}}$ $\boxed{\frac{5}{3}}$

3. Ersetze durch eine Bruchmaschine.

a) $\boxed{\cdot 5}$—$\boxed{:11}$; $\boxed{:9}$—$\boxed{\cdot 4}$; $\boxed{:7}$—$\boxed{\cdot 15}$; $\boxed{\cdot 1}$—$\boxed{:19}$; $\boxed{\cdot 8}$—$\boxed{:12}$; $\boxed{:6}$—$\boxed{\cdot 9}$; $\boxed{\cdot 9}$—$\boxed{:6}$.

* b) $\boxed{:3}$—$\boxed{\cdot 2}$; $\boxed{\cdot 3}$—$\boxed{:2}$; $\boxed{\cdot 7}$—$\boxed{:15}$; $\boxed{:3}$—$\boxed{\cdot 20}$; $\boxed{\cdot 20}$—$\boxed{:3}$; $\boxed{:8}$—$\boxed{\cdot 3}$; $\boxed{:3}$—$\boxed{\cdot 8}$.

4. Ersetze die Bruchmaschine auf zwei Arten durch eine Anlage aus zwei Maschinen.

a) $\boxed{\frac{3}{5}}$; $\boxed{\frac{4}{7}}$; $\boxed{\frac{10}{3}}$; $\boxed{\frac{18}{25}}$; $\boxed{\frac{6}{5}}$. b) $\boxed{\frac{1}{3}}$; $\boxed{\frac{5}{10}}$; $\boxed{\frac{15}{17}}$; $\boxed{\frac{20}{5}}$.

○ **5. Schreibe zu Aufgabe 4 die Operatorgleichungen.**

□ **6. Gib eine 2 cm lange Strecke in die Maschinenanlagen $\boxed{:5}$ $\boxed{\cdot 4}$ und $\boxed{\cdot 4}$ $\boxed{:5}$ sowie in die entsprechende Bruchmaschine. Zeichne wie oben im Beispiel, bei den Maschinenanlagen auch das Zwischenergebnis.**

△ **7. Gib wie in Aufgabe 6 eine 3 cm [24 mm] lange Strecke in die $\boxed{\frac{2}{4}}$-Maschine [$\boxed{\frac{5}{8}}$-Maschine] und in die entsprechenden Anlagen aus zwei Maschinen ein.**

Subsequently during the 1970's new curricula were developed in which various aspects of fractions were presented and combined systematically. This can also be interpreted as the result of an extended concept of mathematical background theory as illustrated in the following table (HAYEN, VOLLRATH, WEIDIG, 1983, p.3):

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Thus, we identify

Phase 4: The whole variety of approaches, properties and aspects of positive rational numbers is considered as background theory for teaching fractions.

This new attitude, in turn, opens up many possibilities for investigations, discussions, and curriculum construction for didacticians.

4. Discussion

From the comparison of the developments of teaching geometry and fractions, it seems obvious that there are similar trends, but with some differences. For example, a rigorous mathematical point of view appears later for fractions than for geometry. Further, a pedagogical tradition is more prominent for fractions than for geometry. An explanation for this might be that the foundation of the number system did not occur before the end of the last century.

We have identified phases in the evolution of an understanding of mathematical theories as background theories. This corresponds to a development of the self-concept of didacticians. But discussions among didacticians continue to reveal different attitudes towards mathematical background theories. Many didacticians are convinced of the important role that certain axiomatic theories play as background theories for mathematics education, but they are not yet ready to subscribe to the wider concept. We must therefore wonder what will be the next phase and weather it will be a step forward or backward. In any case, it is very important in the next phase to identify curricular problems and their related arguments which can be considered with respect to a mathematical background theory in order to get an awareness of the potential as well as the limitations of this approach. Finally, it seems clear that a mathematical background theory is just one of various components of a theory of mathematics education and that its role will also be influenced by changing appraisals of these other components.

Acknowledgments

I thank my friends J.P. Becker for shaping my English and H.G. Steiner for stimulating discussions.

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