26. The understanding of similarity and shape in classifying tasks

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1. Colloquial speech and geometrical terminology

The names of many geometrical concepts are used not only in geometry but also in colloquial speech. But as a result of a process of mathematization, in geometry they are restricted by axioms or definitions. In geometry teaching, names connected with certain images can promote the process of concept attainment in so far as the teacher can use the students' experiences to guide them to a suitable definition. On the other hand, the names of many geometrical concepts arc connected with images which do not necessarily belong to the geometrical concept (e.g., similarity, plane, curve, angle etc.). Therefore these names when used in geometry can hinder the understanding of the concepts by using properties which are not included in the geometrical concept and which may therefore lead to false inferences. A similar situation can be found in problem solving where a word may play a role as "verbal cue" or where the same word can be a "distractor" (JERMAN 1972; NESHER and TEUBAL 1975).

In geometry similarity is a relation between figures (point sets). A figure F_1 is similar to a figure F_2 if there exists a similarity transformation s (composition of a dilatation and an isometry) such that: $s(F_1) = F_2$. lt follows that one rectangle is similar to another if and only if the ratios of the sides are equal. In a deductive procedure this is an inference. Similar results can be found for triangles and polygons. Proofs are given on the basis of the definition using properties of similarity transformations. For a spiral approach to geometry it might be interesting to know whether it is possible to arrive at these results from the basis of a general understanding of similarity in colloquial speech before having learnt a definition.

This could be a basis for a general definition of similarity. For teaching similarity at a higher stage it is necessary for the teacher to know how far images connected with the expression "similar" can support or restrain the learning process for this relation.

Therefore it seems to be useful to know about students' understanding of the expression "similar" as it is used in colloquial speech when it is applied to geometrical figures.

In PIAGET's research on the development of geometrical concepts in children, concepts with names used in colloquial speech play an important role. In his investigations on similarity he shows children of different ages a large rectangle and asks whether it is "similar" to another smaller one, or not. Or he asks whether a given rectangle shown to the children has the same shape as the first, only larger (PIAGET and INHELDER 1948). The terms "similar" and "same shape" are used without a definition just as terms of colloquial speech applied to a mathematical situation. Obviously PIAGET expects certain solutions to a heuristie task, which allow him to argue about the children's concept development. He sees the danger of misleading the children by asking too vague questions; on the other hand he is afraid of giving too narrow guidance by asking too precise questions:

> Il s'agit surtout d'éviter, dans la consigne, l'idée d'une augmentation uni-dimensionelle tout en ne suggérant pas verbalement que l'augmentation doit porter sur le deux dimensions à la fois. (PIAGET and INHELDER 1948, p. 418)

PIAGET changes the method of questioning, the testing materials and the technical aids with the children's age. Therefore his experiments suffer from the fundamental weakness of his experimental design, that he does not isolate the concept development in a domain of perturbing factors (FREUDENTHAL 1973, p. 671).

The understanding of similarity can be tested with classifying tasks, because similarity is an equivalence relation between figures. In mathematical instruction, classifying tasks can make the students conscious of the characterizing properties of the concept, guide them to a definition and control their understanding of the definition. Because of this importance we used classifying tasks to find out students' understanding of ‚similar‘ independent of a mathematical definition.

From the investigations of KAPLAN (1950) one knows about the influence of the verbal context on the development of the understanding of the meaning of unknown words. One might expect that there also is an influence of the "geometrical context" on the results of sorting problems. We therefore studied the influence of the given basic set of figures on the result of the sorting process.

Because of PIAGET's results on the influence of age we explored whether there is a relation between age and the results of sorting problems.

It was reported that there is a difference between boys' and girls' understanding of proportions (MINSKI 1969). We therefore sought a relation between sex and type of solution.

PIAGET used different questions in his tasks. We therefore wanted to know whether there is an influence of the term used on the result of the sorting.

In his experiments on the similarity of rectangles PIAGET used sequences of tasks to guide the Ss to the discovery of proportion. By the investigation of POSTMAN and PAGE (1947) one knows that judgments on one attribute (height or width) of a rectangle in a series of tests effect die judgment of another attribute. We therefore wanted to study the relation between the arrangement of sorting tasks and the results.

Many psychological investigations using sorting problems present a solution as an example, and one wants to find out whether the student can find die relevant attribute (e.g. USNADZE 1929).

Free-object-sorting tests were used by GARDNER (1953) and GARDNER and SCHOEN (1962) in their investigations about cognitive styles in categorizing behavior. They asked Ss "to put together into groups die objects which seem to you to belong together". They measured die different results by the number of resulting equivalence classes. We did not give an example because we wanted to know about the *spontaneous* interpretation of the terms "similar" and "same shape", neither did we ask as generally as GARDNER because we wanted to know about die understanding of the terms "similar" and "same shape".

2. Procedure

In the first experiment a sample of 110 subjects was randomly selected from different schools and different classes in the northern part of Bavaria, FRG. The ages ranged from 8 years to 19 years. The subjects were told that they were taking part in an investigation about the way in which students understand a question in a sorting task. In six tests different collections of paper figures were used, five sets cut out of white paper and one set cut out of checked white paper. In each test all the figures were put on the table unordered (Figures 1-6; scale 3:10).

The subject was asked by die experimenter: „Lege alle ähnlichen Figuren auf einen Haufen!“ (Put all similar figures in a heap!) There were no further comments or aids. The experimenter noted the result, collected the material, and offered die next test. The order of die tests was always the same.

In die second experiment a sample of 50 subjects was randomly selected in die same part of Germany. They got die same tests, but being asked: „Lege alle Figuren mit gleicher Form auf einen Haufen!“ (Put all figures of the same shape in a heap!).

In a third experiment a sample of 30 subjects was randomly selected in the same Part of Germany. They were given only die shapes in Figure 6, and were asked: „Lege alle ähnlichen Figuren auf einen Haufen!“ (Put all similar figures in a heap!).

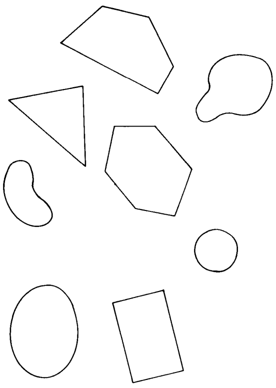

Fig.1

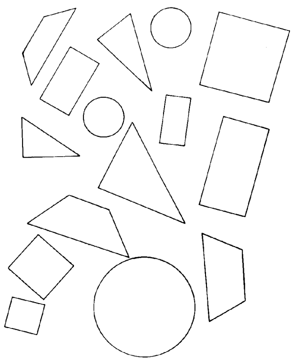

Fig. 2

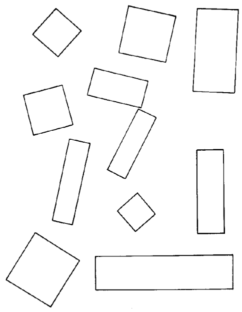

Fig. 3

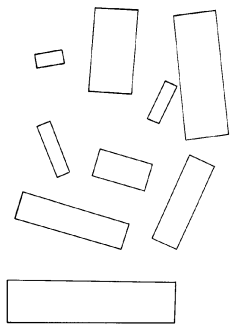

Fig. 4

The experiments were carried out between autumn 1975 and autumn 1976. A. AMEND, A. ROCK, B. KELLNER, and P. SCHMITT, students in mathematical education at the Würzburg University, assisted me as experimenters.

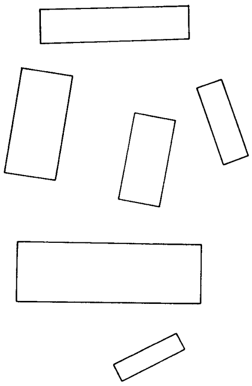

Fig. 5

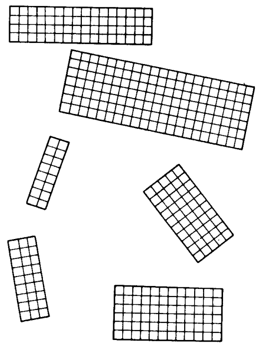

Fig. 6

3. Results

In each experiment a rather large number of different solutions was found in each test. Table I shows die situation in die first experiment. The results were classified by die frequency.

There were two dominant solutions in the first test. A_1 consists of 4 classes containing 2 elements each (Figure 7), whereas B_1 only has 2 classes with 4 elements each (Figure 8).

In both solutions die Ss discriminate between round and angular. Whereas B_1 is restricted to this classification, in A_1 the classification is more refined. The decisive attribute seems to be the *shape* of the boundary line. The somewhat arbitrary separation of the rectangle and die triangle from the pentagon and hexagon in A_1 might be the result of die distinction being made between shapes which are "recognizable" and "many"-sided polygons.

In test 2 the dominating solution results from easily recognized figures such as circles, squares, triangles, rectangles, and trapezia (Figure 9).

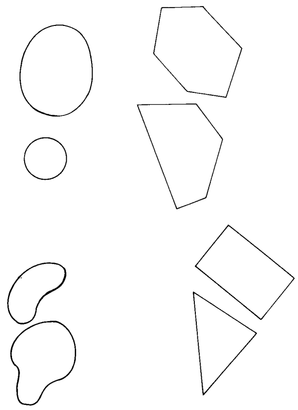

Fig. 7 A_1

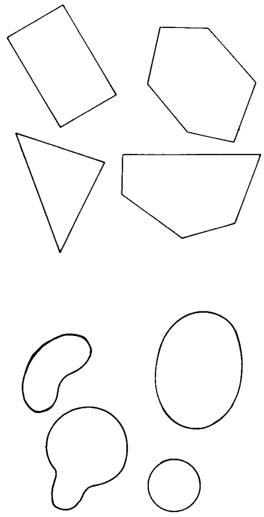

Fig. 8 B_1

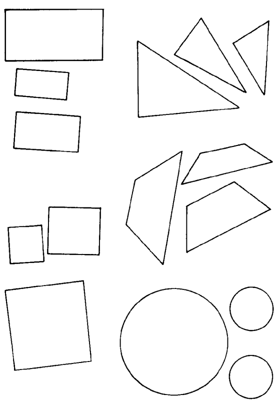

Fig. 9 A_2

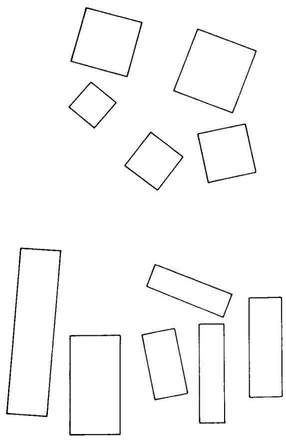

Fig. 10 A_3

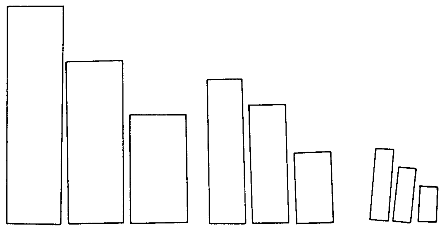

Fig. 11 A_4

It is remarkable that the possible finer classifications among rectangles are not seen, whereas squares and rectangles form different classes.

In test 3 the classification into the 2 classes with squares and rectangles dominates (Figure 10). Possible finer classifications among the rectangles are not seen. The "Gestalt" of the squares influences the discriminating process so that Ss do not see further possible classifications among the rectangles.

For this reason, in test 4 only rectangles were offered (Figure 11). The main solution classifies the rectangles under the aspect of having the same width. Obviously the attribute "having the same width" dominates the attribute "having the same side ratio". Another frequent solution keeps all rectangles together. The Ss seem to see no corresponding attributes which allow a sorting related to "similar". The result might also be influenced by the preceding test.

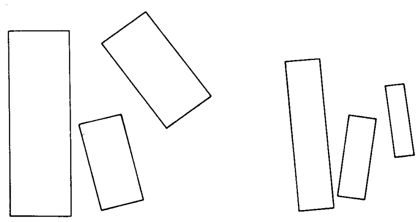

Fig. 12 A_5

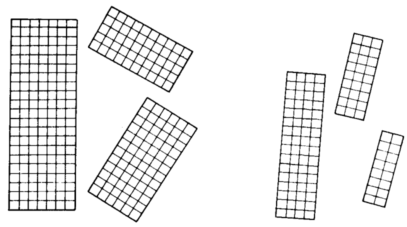

Fig. 13 A_6

In tests 5 and 6 (Figures 12 and 13) rectangles were offered which can be classified into 3 classes, when one considers the side-length ratio. But no two of them arc of the same width or length. There was no solution among the whole sample which took account of the attribute of having the same sidelength ratio. Thus no one discovered mathematical similarity as a principle of classifying, even though the only ratios used, such as 1:2, 1:3, and 1:4 gave a strong hint. Even the squares drawn on the rectangles of test 6 were of no assistance. In both tests sorting relative to a vague idea of size (large, small) dominated.

Table I

Frequency of solutions

Test	Solution		Frequency
1	18 different solutions		
	A_1:	(Fig. 7)	27%
	B_1:	(Fig. 8)	26%
	C_1: (further different solutions; at most 10% each)		47%
2	13 different solutions		
	A_2:	(Fig. 9)	77%
	B_2: (further different solutions; less than 10% each)		23%
3	15 different solutions		
	A_3:	(Fig. 10)	66%
	B_3: (further different solutions; less than 13% each)		34%
4	20 different solutions		
	A_4:	(Fig. 11)	30%
	B_4:	rectangles	26%
	C_4: (further different solutions; less than 10% each)		44%
5	10 different solutions		
	A_5:	(Fig. 12)	51%
	B_5: (further different solutions; less than 16% each)		49%
6	11 different solutions		
	A_6:	(Fig. 13)	62%
	B_6: (further different solutions; at most 15% each)		38%

To investigate the relation between kind of solution and age/sex, chi-square tests were used, which arc summarized in Table II. Table II presents the number of Ss for each type of solution by sex and by age for each test.

Table II

Comparison on solution type by sex and age

		Sex		Age			
Test	Solution	M	F	≤10	12	13,14	≥15
1	A_1	16	14	5	9	9	7
	B_1	21	8	6	7	9	7
	C_1	27	24	15	18	12	6
		$\chi2 = 3.28$ df= 2 $p<0.20$		$\chi2 = 4.86$ df = 6 $p<0.70$			
2	A_2	41	31	17	27	25	16
	B_2	23	15	9	7	5	4
		$\chi2 = 0.04$ df = 1 $p<0.90$		$\chi2 = 2.89$ df =3 $p<0.50$			
3	A_3	41	31	9	25	24	16
	B_3	23	15	17	9	6	6
		$\chi2 = 0.13$ df = 1 $p<0.80$		$\chi2 = 14.93$ df =3 $p<0.01$			
4	A_4	22	11	8	9	8	8
	B_4	19	10	3	11	9	6
	C_4	23	25	15	14	13	6
		$\chi2 =3.70$ df = 2 $p<0.20$		$\chi2 = 5.89$ df =6 $p<0.50$			
5	A_5	32	24	15	14	15	12
	B_5	32	22	11	20	15	8
		$\chi2 = 0.05$ df = 1 $p<0.90$		$\chi2 = 2.44$ df =3 $p<0.50$			
6	A_6	39	39	18	23	14	13
	B_6	25	17	8	11	16	7
		$\chi2 = 0.05$ df = 1 $p<0.90$		$\chi2 = 4.10$ df =3 $p<0.30$			

Chi-square tests show that in test 3 a relation between age and type of solution can be assumed ($p<0.01$). With increasing age more Ss prefer solution A_3.

Table III

Comparison on type of solution by type of request

Test	Solution	Request "similar"	"shape"
1	A_1	30	21
	B_1	29	11
	C_1	51	18
	$\chi2 = 3.45$ df = 2 $p<0.30$		
2	A_2	85	38
	B_2	25	12
	$\chi2 = 0.03$ df = 1 $p<0.90$		
3	A_3	72	41
	B_3	38	9
	$\chi2 = 4.54$ df = 1 $p<0.05$		
4	A_4	33	17
	B_4	29	28
	C_4	48	5
	$\chi2 = 20.39$ df = 2 $p<0.001$		
5	A_5	56	26
	B_5	54	24
	$\chi2 = 0.02$ df = 1 $p<0.90$		
6	A_6	68	35
	B_6	42	15
	$\chi2 = 1.00$ df = 1 $p<0.50$		

The different solutions for the six tests in the second experiment are named as in the first experiment (Table I). The results of the second experiment are listed in Table III. They are combined with the corresponding data from the first experiment.

Chi-square tests for the relation between type of solution and type of request in the second experiment are listed in Table III.

These show significant relations between type of solution and type of request in test 3 ($p<0.05$), and in test 4 ($p<0.001$). If the question uses the word ‚shape' the Ss prefer the solution with 2 classes (rectangles and squares) in test 3. In test 4 most Ss prefer the solution of all rectangles forming only one class when asked for the same shape, whereas the other group of Ss has a majority for dividing into classes with the same width. For tests 1, 2, 5 and 6 the hypothesis that there is no difference between the results in the two groups cannot be rejected. Therefore one can assume that varying the question in the manner tested does not lead to a better solution with respect to geometrical similarity of rectangles.

There were two dominant solutions in the third experiment when only the rectangles of test 6 were presented to the Ss: Again A_6 (23%), but a solution B_6' (47%) consisting of three classes containing 2 rectangles each (Figure 14). The "correct" solution was not found.

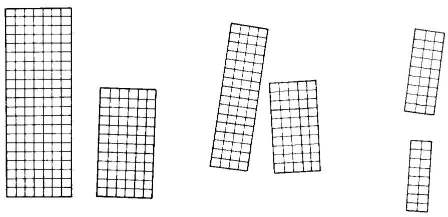

Fig. 14 B'_6

The Chi-square test for the relation between the type of solution and the arrangement of the tests for test 6 in the third experiment, is presented in Table IV.

Table IV

Comparison on type of solution by

type of arrangement

	Arrangement	
Solution	Sequence	Isolated
A_6	68	7
B'_6	9	14
C'_6	33	9
	χ^2=27.77 df =2 p<0.001	

This shows a significant influence of the preceding tests on the solution of test 6. An isolated presentation seems to tend more to differentiation among the rectangles.

4. Discussion

In each test relatively many different solutions were found. The request to put together similar figures is understood in many different ways. Obviously the Euclidean solution is not being noticed. The wide range of different solutions in all sorting tests might be explained by the personal styles of experiencing (THOULESS 1932; WEBER 1939; GARDNER 1953).

Sex and age, in the age group investigated, do not seem to influence the solution, whereas the type of question asked in the sorting task seems to have some influence. Although "similar" in the Euclidean sense and, "of the same shape" indicate the same geometrical relation, they are understood in different ways when used in sorting problems. "Similar" seems to tend more to attributes of the figure, whereas "same shape" tends more to the "Gestalt".

Another possible interpretation might be that "of the same shape" is understood as "described by the same shape-name". This might be the result of instruction since younger children are asked the "same shape" question in mathematics education specifically to acquire a "shape-name" vocabulary. The performance of the sorting task is influenced by the importance attached to different aspects of the dimensions of the figure. It is known that, e.g., color is more obvious to children than shape (USNADZE, 1929). In general ARCHER (1962) found that increasing the obviousness of relevant attributes facilitates the learning of a concept, whereas increasing the obviousness of irrelevant attributes impedes it. In our tests one can see that "equal width" is more obvious than "equal ratio of sides" for rectangles. In general, the side-length ratio seems to be rather inconspicuous for rectangles. This explains the poor results in test 5 and test 6.

The influence of the chosen basic set of figures can be explained by the importance of the "frame of reference" (KOFFKA 1935; WITTE 1960).

The responses in series of sorting tasks are effected by the order of presentation as reported by HALL (1950) for the naming of drawings, and by IMMERGLUCK (1952) for judging pairs of geometrical designs for symmetry. This may be understood as developing "Einstellungen" during the performance of a task. The effect of repeated activities may also be explained by a process of reinforcement. The S is then "blinded" to new possibilities (LUCHINS 1942).

5. Implications for teaching strategies

Teaching strategies for geometrical concepts which also have a meaning in colloquial speech should take into consideration the fact that the mere knowledge of the name does not include the full geometrical understanding of the concept. Therefore it is necessary to standardize the understanding within the class by stating a definition. General experiences can be a substratum of this definition, but its understanding should be controlled by tests including critical cases.

For teaching the concept of similarity it seems to be natural to start with similarity transformations. Then It is an interesting problem to find criteria for the different polygons to be similar. In such a program all problems are well defined, so that the students can understand what is wanted from them (contrary to PIAGET's experiments). Sorting problems can help to test students' understanding of the definition and can guide them to assumptions about properties of similar figures. Further help can be given by emphasizing inconspicuous attributes by choosing extreme cases, or by coloring relevant parts of the figures considered. The discussion about the concept of similarity should include the understanding that "having the same shape" and "being similar" are equivalent.

Similarity of rectangles can be expressed in terms of proportions. Therefore a relation between the understanding of similarity in geometry and proportionality in algebra might be supposed. LUNZER and PUMPHREY (1966) found that there is a preference of additive structures over multiplicative in solving number problems. ABRAMOWITZ (1974) studied the ability of solving proportions under the aspect of understanding rules for calculating fractions. FREUDENTHAL (1974) discussed proportions under the aspect of functions. These investigations suggest that one must be careful not to generalize too quickly from results on any similarity task to the concept of similarity in general.

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