

24. The place of geometry in mathematics teaching: An analysis of recent developments

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I shall start from the assumption that, nowadays, almost everyone needs some kind of geometric knowledge and abilities. Therefore, in discussions about geometry, didacticians should consider the great variety of persons to whom geometry must be taught.

Recent developments show the necessity to evolve an individualized curriculum which takes into account the various needs and abilities of the students.

1. THESIS: *In the development of an individualized curriculum the needs and abilities of the students should be given priority over the interests and preferences of teachers.*

This statement should be self-evident. However, its formulation becomes necessary when analysing didactical proposals concerning the teaching of geometry. We will all admit, I think, that the proposal of teaching trigonometry in the Poincaré-model of non-Euclidean geometry to high school students is an example of possible dominating interests of teachers.

Our experience shows – especially in the field of geometry – that enthusiasm for a theory, a special liking for tricky problems, fanatic rigor, or sophisticated philosophical ideas may blind the teacher to the needs and abilities of the students.

As was pointed out by KAUFMAN at the Carbondale Conference, one of the primary goals of CSMP was to develop an individualized curriculum for students aged between 5 and 18. Curricula should be developed, "which provide for each student a program sound in content, enjoyable, and most appropriate for his needs and abilities" (KAUFMAN 1971, 2). This statement will underlie my further considerations.

In my opinion, CSMP tried to evolve an individualized curriculum by follow-

ing different approaches such as topology, congruence-geometry, linear algebra, projective geometry, combinatorial geometry, and general activities such as mathematization, algebraization, experimental approaches, and the use of the geometrical language. In Germany, the teaching of geometry proceeded from the same idea: One starts with an experimental, intuitive course in the lower grades and goes on in the intermediate grades with courses in subject areas of geometry centered on one of the mentioned theories, thus teaching in a more or less deductive way. Didactical discussions have mainly been concerned with finding the most suitable mathematical theory to serve as a background. This is not a very fruitful starting point for developing an individualized curriculum.

2. THESIS: *An individualized curriculum which satisfies the needs and abilities of students cannot be designed by varying the geometrical theory as an organizing principle.*

We can agree with KAUFMAN when he says:

"To have a truly individualized curriculum, one must take into account the different views of mathematics held by various people ... Each has needs that some form of mathematics can fulfil. Each can enhance his services to society by bringing to his vocation an 'appropriate' background in mathematics" (KAUFMAN 1971, 2-3).

Although the necessity of teaching geometrization and emphasizing creative and problem-solving aspects is seen (SCHUSTER 1971), more or less system-oriented courses are still predominant in high-school education. This is due, I feel, to the failure to develop convincing principles for organizing teaching in a fashion suitable for the different aspects of geometry.

3. THESIS: *Individualized curricula can be developed by varying the guidelines according to the different views of geometry.*

I would like to point out different views of geometry and develop for each of them adequate guidelines for designing individualized curricula.

4. THESIS: *The teaching of geometry to all students in mathematical education can only be justified if we succeed in evolving differentiated curricula for geometry teaching.*

1. Geometry as an inventory of mathematical theories

This is a point of view from which many mathematics teachers who give priority to the receptive aspect see geometry. Geometry has been taught in this perspective for generations without it being questioned. The other aspects such as applications (e.g. ballistics, etc.), problem solving, and the properties of space (e.g. optics, mechanics etc.) were subordinated to this aspect. Many proposals have been put forward in the last years to use different geometrical theories for teaching purposes. These attempts were focussed on:

formal goals: Which theory can best account for mathematical thinking. *content-oriented goals:* What content is most important in geometry? *students abilities:* Which geometrical theory allows the easiest approach to geometry?

Among the numerous competing proposals I would like to point out some effective and well-tested ones that seem important for future teaching.

The ideas of transformation-geometry have proved effective for grades 7 and 8. In Germany, mapping-oriented courses were developed and practiced after the proposals made by WILLERS (1962), FLADT (1956), JEGER (1964) and FABER (1958).

The discussions about founding geometry on the principles of reflection (WILLERS 1962; FABER 1958) or competing, on affine mappings (PRADE 1966) seem largely geared to the personal preferences of geometers.

I think that JEGER's Constructive Transformation-Geometry and STEINER's proposal at the Carbondale Conference (STEINER 1971) to make use of congruence mappings as a foundation for geometry aptly summarize the ideas of

transformation-geometry. The concept of group has become a guideline for some fruitful activities as was recommended by the OEEC 'SYNOPSIS' and the Carbondale Conference. Convincing examples of applications of the ideas of the 'Erlanger Programme' are provided by the systematology of the quadrangle by means of symmetry groups (BAUERSFELD 1961/1962) and DIENES' activities.

A broad discussion was brought on by the conception of linear algebra. DIEUDONNÉ (1969), CHOQUET (1970), SERVAIS (1967) and PAPY (1965) made valuable contributions to this approach for intermediate grades. PICKERT used the conception of linear algebra for axiomatization at the Aarhus Conference. It seems adequate for teaching classes from grade 11 upward.

Special geometries such as inversive geometry (COXETER 1971), geometry of Polygons (BACHMANN 1971), incidence geometry (ZEITLER 1970), and finite geometry seem mainly to serve as a formal training to mathematical ways of thinking. When evaluating these approaches, one should consider the precious short time the students have at their disposal. It should be questioned whether these goals could not be equally well reached by activities centered on more important contents.

Finally, many proposals have been made for making use of topological ideas for teaching geometry. I refer to HILTON, KELLY, PAPY (1971), WALLRABENSTEIN (1973), and WEIDIG (1970). These have been successfully practised as a problem-oriented introduction at the primary level, partly proceeding from PIAGET's theory of the development of geometrical thinking.

All these approaches tend to teach geometry as a ready-made product derived from a background theory.

2. Geometry as an inventory of concepts and theorems for constructing theories

This is a point of view from which mathematicians who stress the creative aspects see geometry. The criticism of an instruction conceiving of mathematics as a ready-made product led to new organizing principles. With the idea of genetic teaching (for a more detailed exposition: cf. WITTMANN 1974,) KLEIN (1908), TOEPLITZ (1949), and WAGENSCHN (1961) made important suggestions. This principle allows a fruitful instruction

focussed on axiomatization

(e.g. FLADT's 'Parallelenlehre', 1952). PICKERT gave an example of plane geometry at the Aarhus Conference.

FREUDENTHAL's idea of 'local ordering' has stimulated the didactical discussion. At the Oberwolfach Conference in 1962, GRIESEL, KIRSCH, and STEINER made interesting proposals for axiomatization exercises.

Important activities in this connection are: The definition of concepts, the search for primitive concepts, the formulation of theorems, demonstrations, proof analysis, the search for logical relations, the comparison of different definitions of a concept, the compilation of an inventory of theorems, generalization and the discussion of methods.

But there is the danger that teachers use the idea of 'local ordering' as a justification for presenting a locally ordered geometry as a ready-made product.

3. Geometry as an inventory of strategies for solving problems

This is another point of view from which creativity-oriented mathematicians look at geometry. The stimulating power of geometrical problems has also caused non-mathematicians to become interested in mathematics. In the pedagogical discussion, problem-oriented teaching has become very important.

Problems are used for motivation and application purposes and as central subjects of instruction.

Analysing proposals for a problem-oriented geometry teaching, one mainly finds courses

focussing on a mathematical background theory.

Problems are used to lead to the discovery of theorems which provide solving strategies for further problems. Obviously, this idea does not make for a substantial new approach.

Traditional instruction made use of collections of similar problems such as triangle constructions, geometric loci. These were

centered on the systematics of the subject

(e.g. variation of the given elements). As we know, this is a very important formal principle: It may serve to facilitate recollection and help finding a red thread. But I think that it has been somewhat overstretched in traditional instruction. It may happen that problems that are more difficult to solve than later ones are treated first. Therefore, this orientation may turn out to be more of a limitation than a help.

Recent proposals concerning stimulating problem sequences (WAGENSCHNEIDER 1961, WITTENBERG 1963; ENGEL 1971) are often

centered on a subject area (WITTENBERG),

(e.g. optimization (ENGEL), subdivision of the plane (ENGEL), and similarity (WITTENBERG)). For the discussion on this contribution, refer to WITTMANN. I consider this idea of subject area to be a fruitful orientation for a geometry course in which problems dominate.

If such a subject area is related to an every-day project – business, or science – we obtain a sequence of problems

focussed on a project

(e.g. tiling, colouring of maps (DYNKIN, USPENSKI 1955)). These activities appear in many curricula nowadays.

From the experience gathered in courses in which students are trained in order to sit a final examination or a mathematical competition (Olympiad), problem sequences have been developed that are

centered on solving strategies

which operate with a growing degree of difficulty. This principle has overrun traditional algebra instruction (LENNÉ 1969). I think it is useful for local organization only.

An interesting problem-oriented instruction can be provided by courses

focussing on games.

A stimulating example is supplied by SCHUPP's "nine men's morris-geometry". He compiles a collection of problems by considering the nine men's morris-configuration as a finite geometry.

Problem-oriented instruction is mainly motivated by the interest in formal goals. Important activities are: The transformation of a problem, the search for similar problems whose solutions are known, the finding of an assumption, the development of solving strategies for several types of problems, the comparison of different solutions, and the development of new problems through problem solving (WITTMANN).

Problem-oriented teaching can further creative thinking, and motivate for mathematics in general. But one must also see the possible discouraging effect of difficult problems on unsuccessful students, or the stupefying effect of mere exercises on gifted learners.

4. Geometry as an inventory of theories for action

Many people need geometry as a background for technical and constructional purposes, or for manipulating instruments, to make decisions, etc. Even for these students, geometry has been taught in a more or less deductive, system-oriented way. Practical problems or activities are only used to motivate and provide applications. I think teachers are too fixed upon their mathematical way of thinking. It should be our goal to devise ways to achieve a better integration of practice. One of these ways seems to be the development of learning sequences

geared to the use of instruments.

The classical problem of the constructions possible with a pair of compasses and a straight-edge led to hairsplitting, as a result of permitting only these instruments to be used. But it seems natural to start by exploring the possibilities of these instruments. Another method is to consider the constructions possible with an architect's drawing machine and to try to explain how it works.

Other guidelines such as the question of the constructions possible with a mirror-ruler (PROKSCH 1956) or a parallel-ruler (PRADE 1966) can supply different geometrical theories of the plane.

Lessons can also center on special types of instruments such as lengthmeasuring instruments (explanation of the Vernier scale).

For professional education, it may prove useful to design units of instruction

centered on a project,

(e.g. the drawing of a floor by an architect. I refer to FLETCHER's Architect's Geometry).

Another example may be supplied by road building: The finding of the shortest way between two villages, of the point with the minimal distance sum to three other points, the construction of curves and tangents, the planning of curves

allowing certain maximum speeds etc. These activities can readily lead from elementary to difficult questions.

Some time ago the news told us that all our maps were ideological, because underdeveloped parts of the world were too small on these maps in relation to developed parts. It was announced that a realistic map had now been drawn. Such a statement can be a motivation for studying cartography questions. It is easy to develop different types of projections from a sphere on a plane. They may first be described constructively. Questions concerning the invariants of the used mappings such as area, length, and angle may be answered through elementary considerations for some important types. These problems may then be discussed at a higher level by means of analytic methods. The necessary tools are made available by an introduction of parametric representations of the sphere, the plane, and mappings. In a project, we could start with the drawing of a city map (similarity questions), which could lead to the construction of a world map. The subject area of map-colouring can be very fruitfully integrated.

Another example of this is the study of roof construction. The different kinds of roofs used in house and church building throughout Europe can be analysed in terms of the observed surfaces and solids, beginning with plane surfaces and going on to ruled surfaces, which can also be seen in modern architecture. The project can be extended to cover roof statics, which can provide an interesting access to vectors.

Applications of geometry to economics such as linear optimization, or the methods of linear algebra in the theory of the market may be useful for courses

centered on aids to decision making.

The mentioned theories should not be used as applications but as the results of mathematization. PAPY's experiments with 'shopping lists' resulting in vector spaces show a possibility of teaching linear algebra in a practice-oriented way. I see a great potential of motivations in the nearness of practice for practice-oriented students, a chance to prepare them for their future professional life

and for practising an activity-oriented teaching. On the other hand, we should see the danger of remaining at the merely technical level without the geometrical ideas being understood.

5. Geometry as an inventory of theories of space

FREUDENTHAL and WAGENSCHIEIN have pointed out that students need fundamental geometrical insights into space. WAGENSCHIEIN recalls the Greek idea of geometry as "mathematics out of the earth". FREUDENTHAL quotes many examples of fundamental stimulating questions such as

"Why does a piece of paper fold along a straight line?

Why is the straight line the shortest?

What is the relation between the real and the apparent size of a body;

What is a rigid motion on the sphere?" (FREUDENTHAL 1973, 404).

Possible learning sequences may be

centered on mathematizing the environment of the students.

The study of geometrical objects and their interrelationships in the classroom and the performance of experiments with solids – to have the students gain experience with shapes, surfaces and solids – are practised in all kinds of modern elementary mathematics instruction. Pioneers were TREUTLEIN (1911) and VAN HIELE (1957). However, not only the more or less technical environment of children should be analysed. As WAGENSCHIEIN recommended, we should try to understand our natural environment, too, by means of geometry. I realized the necessity to do so when I witnessed the experience of a boy from Berlin for the first time seeing the night sky from the top of a mountain in the Odenwald. He was only accustomed to the sky signs of the city.

The study of nature can be achieved by courses

centered on physical experiences.

The most important physical theories resorting to geometry are mechanics, optics, and the theory of relativity. In physics, geometry is applied to observations as a fitting theory. For geometry teaching, it is important to derive a theory from a geometrization of nature. This approach may be different from that of physics teachers and we cannot expect physics teachers to teach the process of the mathematization of nature.

These questions can lead to more general problems of space, which might be interesting for students concerned with philosophical questions. Therefore, I want to point out courses

focussing on the problem of space.

The discussions about teaching non-Euclidean geometry refer, besides the idea of axiomatization, to the ideal of achieving competing models of reality. I think the approaches based on the Klein- or Poincaré-models are not very fruitful, being too remote from reality. The more fruitful approach might be provided by the Minkowski-geometries as recommended by LAUGWITZ in 1958. PICKERT's approach to a metric through the conics at the Carbondale Conference may also lead to such considerations.

Finally, I would like to state that it is precisely at a time when space travel is topical that spheric geometry has sunk into oblivion in mathematical education. My point is not to restore the excessive teaching of spheric trigonometry, but to suggest that our traditional good relation to astronomy should be cultivated.

A geometrical course organized from the point of view of space theory can make important contributions to a higher education which enables students to better understand their environment and form a modern conception of the world. It may convince future non-mathematicians that geometry is an important tool for describing and understanding nature. So they might derive motivations and aids for illustration purposes. But this approach may also prove a hindrance to further studies in geometry which are not so close to questions

related to nature. Another well-known danger is that the experiments have such persuasive power that the need for a proof is often overlooked.

6. Geometry as the result of solved problems

This is a point of view of specialists interested in intellectual history. In discussions about genetic teaching the idea was formed to develop courses

centered on the historical development of a problem and its solution.

TOEPLITZ recommended this as a 'direct genetic method'. He hoped to manage to motivate people for mathematics who had been frustrated by mathematics presented in an uninspiring fashion. Good examples of fruitful applications of this idea are: The development of the problem of the constructions possible with a pair of compasses and a straight-edge, or the determination of a surface (TOEPLITZ 1949).

A course may also be

focussed on the contributions of an important epoch of man's culture.

Examples of this approach are the theory of proportions, the space problem in the 19th century. These orientations give the opportunity to try to establish social and philosophical relations.

For students at a higher level I could imagine studies

centered on the geometrical contributions of a famous mathematician

Examples are DESCARTES, RIEMANN, HILBERT, and KLEIN.

In this perspective, typical activities would be: The reading of an original text, the translation of old phrases into a modern mathematical language, the reading of comments, the analysis of mathematical, philosophical, or social relationships, of different solutions, the discussion of varying formulations of a problem, the discussion of a problem from a modern point of view, the investi-

gation of the influence of special problem formulations on recent research.

The teacher may introduce mathematics as a result of efforts to solve problems, so that mathematics appears as a living science. From a social point of view, mathematics can be studied as the result of research at a specific time in a specific environment. But one must see that this is a tedious method, which is neither fruitful for all geometries nor for all students. Furthermore, it has not yet been satisfactorily developed as a teaching strategy.

7. Geometry as an inventory of forms

We know of geometric studies by famous artists in history. These examples show that we should also see the need for art-oriented courses in geometry. These can be

centered on the observation and analysis of forms

in nature (nets, tissues, crystals, etc.), in architecture (e.g. symmetry of gothic churches), and arts (e.g. ornaments). I would like to recall the historic discussions about the role of the golden section and about perspective drawing. Questions concerning descriptions, the analysis and the generation of forms make for an interesting instruction. From primary education onwards children should become familiar with the variety of geometric forms and relations. And, as can be seen, most modern elementary courses try to do this. But we should also try to make these ideas fruitful for students at a higher level.

The above reviewed conceptions of geometry are surely not exhaustive but I hope that I have considered the most relevant for didactical discussions.

It should be difficult to associate unequivocally every activity used in geometry teaching with a special conception of geometry – which is not so much the point as the furtherance of the search for new activities.

Some general principles for organizing mathematics education have been

quoted. As far as I can see, these are consistent with my reflections.

This exposition of the different possibilities of organization is meant to help the planning of geometrical education.

For global planning:

We could emphasize a special view of geometry in a curriculum for a special group of students.

For local planning:

We could arrange different activities related to different views of geometry in a learning sequence for a special group of students.

Remarks

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