9. Some algebraic aspects in analysis teaching

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It is one objective of teaching analysis in the secondary school to enable the students to use the rules correctly. Another aim is to teach them the foundations of the theory. But unfortunately the proofs are often too difficult as to be found by the students themselves or even to be understood by them on an elementary level. Therefore we must look for selected topics which can be approached by the students mainly on their own. These topics should be important enough for the efforts of the students.

We show in an example how algebraic methods can give the students an idea of a small theory which they can find by themselves and which is relevant to a larger part of analysis.

1. The derivative of a polynomial function

The first class of functions discussed in analysis teaching is the set F of polynomial functions

$$f\colon x \longrightarrow a_n x^n + \ldots + a, x + a, \; x \in \mathbf{R},\; a_k \in \mathbf{R},\; k = 0, \ldots, n.$$

Because we want to consider such functions as algebraic objects we define sum and product:

$$f + g\colon\; x \longrightarrow f(x) + g(x),$$

$$f \cdot g\colon\; x \longrightarrow f(x) \cdot g(x).$$

By these definitions $(F, +, \cdot)$ forms an integral domain with $\tilde{0} : x \longrightarrow 0$ as additive neutral element and $\tilde{1} : x \longrightarrow 1$ as multiplicative neutral element. Now f can be written as

(A) $$f = \tilde{a}_n I^n + \ldots + \tilde{a}_1 I + \tilde{a}_1,$$

with $I: x \longrightarrow x$ and $\tilde{a} : x \longrightarrow a$ for any $a \in \mathbf{R}$. Let the derivative Df of f be found as

$$Df : x \longrightarrow na_n x^{n-1} + ... + 2a_2 x + a_1,$$

or

(B) $$Df = n\tilde{a}_n I^{n-1} + \ldots + 2\tilde{a}_2 I + \tilde{a}_1$$

This can be done by calculating

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

or in a pure algebraic way:

$$f(x + h) \equiv f(x) + Df(x) - h \pmod{h^2}$$

by congruences modulo h^2 (LEVI 1968).

D is a mapping of F into F. The students get the problem to characterize D algebraically.

2. Algebraic characterization of the derivative

It is a typical problem of modern mathematics to characterize an object by its general attributes. It is the beginning generalization of a mathematical method.

To find a solution we can transform the problem into the question: Which rules have we to know for the derivatives of the 'building stones' of f to be able to find Df ?

We show a possible way of answering that question. A first catalogue of axioms may be found as follows.

f is a sum of functions. We should know $D(f + g)$. Comparing (A) and (B) we find

(I) $D(f + g) = Df + Dg.$

The sum (A) consists of products of type $\tilde{a}f$. What can we say about $D(\tilde{a}f)$?

We find again by comparison

(II) $D(\tilde{a}f) = \tilde{a}\, Df$.

We should also know D(I'). (A) and (B) give us

(III) $D(I^n) = nI^{n-1}$

Are we ready now? Since (III) is only defined for $n \geq 1$ we did not get $D\tilde{1}$.

We postulate

(IV) $D\tilde{1} = \tilde{0}$.

We now show that (I)-(IV) is a complete system of axioms for D. We conclude

$$D\tilde{a} = D(\tilde{a}\cdot\tilde{1}) \underset{(II)}{=} \tilde{a}D\tilde{1} \underset{(IV)}{=} \tilde{a}\cdot\tilde{0} = \tilde{0},$$

and by induction

$$D(f_1 + \quad + f_n) = Df_1 + \text{- -.} + Df_n.$$

This delivers (B).

But we now discuss our system of axioms. (III) is not simple enough, because it can be built up by the 'building stone' I. We can find a new system of axioms for D by asking the question: What should we know to be able to prove (III) by induction? For the first step of the induction we need DI. We find by (A) and (B)

(V) $DI = \tilde{1}$.

For the second step of the induction we must be able to express D(If). We find by (A)

$$If = \tilde{a}_n I^{n+1} + \ldots + \tilde{a}_1 I^2 + \tilde{a}_1 I,$$

by (B)

$$\begin{aligned} D(If) &= (n+1)\tilde{a}_n I^n + \ldots + 2\tilde{a}_1 I + \tilde{a}_0 \\ &= n\tilde{a}_n I^n + \ldots + \tilde{a}_1 I + \tilde{a}_n I^n + \ldots + \tilde{a}_1 I + \tilde{a}_0 \\ &= Idf + f. \end{aligned}$$

We therefore demand

(IV) $D(If) = f + Idf.$

We can replace (III) by (V) and (VI). But this is too much. We only need (V1), because, if we take as second system

(I) $D(f + g) = Df + Dg,$

(II) $D(\tilde{a}f) = \tilde{a}Df,$

(IV) $D\tilde{1} = \tilde{0}$,

(VI) $D(If) = f + IDf,$

we can prove

$$DI = D(I \cdot \tilde{1}) \underset{(VI)}{=} \tilde{1} + ID\tilde{1} \underset{(IV)}{=} \tilde{1} + I \cdot \tilde{0} = \tilde{1},$$

and this is (V). Again from (II) and (IV) we get $D\tilde{a} = \tilde{0}$, and by induction we now can prove (III): $D(I^n) = nI^{n-1}$. Another induction delivers again

$$D(f_1+\ldots+f_n) = Df_1+\ldots+Df_n.$$

So we find (B). The derivative is again characterized by an algebraic system of axioms.

(I) and (II) show that D is an endomorphism of the module F over **R.** Is it also an endomorphism concerning the multiplication? We should know $D(fg)$. By generalizing (VI) we find the supposition

$$\text{(VII)} \quad D(f \cdot g) = Df \cdot g + f \cdot Dg$$

and prove it. This gives a third system of axioms for D:

(I) $D(f+g) = Df + Dg$,

(II) $D(\tilde{a} f) = \tilde{a} Df$,

(V) $DI = \tilde{1}$

(VII) $D(f \cdot g) = Df \cdot g + f \cdot Dg$

We can now prove $D\tilde{1} = \tilde{0}$ because

$$Df = D(f \cdot \tilde{1}) \underset{(VII)}{=} Df \cdot \tilde{1} + f \cdot D\tilde{1}, \text{also } \tilde{0} = f \cdot D\tilde{1}.$$

If $f \neq \tilde{0}$ we get $D\tilde{1} = \tilde{0}$, this is (IV), since F is integral domain. Again by (II) and (IV) we get $D\tilde{a} = \tilde{0}$ and by induction

$$D(I^n) = nI^{n-1} \text{ and } D(f_1+\ldots+f_n) = Df_1 + \ldots + Df_n.$$

Therefore D is again characterized by a system of algebraic axioms. From the algebraic point of view D is the derivation (KUROŠ 1964) of F with $DI = \tilde{1}$.

3. The problem of inversion of a mapping

Does there exist an inverse mapping to the mapping *D* of *F* into *F*? Evidently not, because, e.g., $D(I + \tilde{a}) = \tilde{1} = DI$. Therefore it happens that two different functions have the same image under the mapping *D*. We now want to know all functionsf with the same image *Df*. We define:

$$f \sim g \underset{def}{\Leftrightarrow} Df = Dg$$

is an equivalence relation given by the homomorphism *D*. The class of functions *g* equivalent to *f* is

$$\bar{f} = \{g \,|\, Df = Dg\} \ .$$

We want to know more about this class:

$$Dg = Df \Rightarrow Dg + (-Df) = Dg + D(-f) = D(g - f) = \tilde{0} \ .$$

We need the set of all functions mapped on $\tilde{0}$. Let

$$g = \tilde{c}_m I^m + \ldots + \tilde{c}_1 I + \tilde{c}_0$$

be an arbitrary function with $Dg = \tilde{0}$. Then

$$\tilde{0} = Dg = m\,\tilde{c}_m I^{m-1} + \ldots + \tilde{c}_1 \ .$$

Therefore $\tilde{c}_m = \tilde{c}_{m-1} = \ldots = \tilde{c}_1 = \tilde{0}$. We already know $D\tilde{c}_0 = \tilde{0}$. The set of all functions mapped on $\tilde{0}$ is therefore $\left\{\tilde{a} \,\middle|\, a \in \mathrm{R}\right\}$, the *kernel* of the homomorphism (KUROŠ 1964) . Every class *f* can be written as

$$\bar{f} = \left\{f + \tilde{a} \,\middle|\, a \in \mathrm{R}\right\} .$$

Let $\overline{F} = \left\{ \bar{f} \middle| f \in F \right\}$. The mapping

$$\delta: \overline{F} \to F \text{ with } d(\bar{f}) = Df$$

is an isomorphism. This illustrates the *Homomorphism Theorem.* Our algebraic consideration makes clear the connection between derivative and indefinite integral. $\bar{f}$ consists of all indefinite integrals of *f*.

4. The extension of a mapping

Another typical algebraic problem is to find an extension of a mapping to a larger domain. In our case, we ask for an extension of *D* to the field of rational functions, so that the extension is again a derivation.

Let $\overline{D}$ be an extension of *D*, then for *f*, $g \in F$, $g \neq \tilde{0}$, we find

$$\overline{D}f = \overline{D}((f/g)g) = \overline{D}(f/g)g + (f/g)\overline{D}g.$$

Multiplication by *g*,

$$g\overline{D}f = \overline{D}(f/g)g^2 + f\overline{D}g,$$

this delivers

$$\tilde{D}(f/g) = \frac{\tilde{D}fg - f\tilde{D}g}{g^2}.$$

With $\tilde{D}f = Df$ for all $f \in F$ we have a derivation which is an extension of *D*. So we have found in a pure algebraic way the rule for the derivative of a quotient.

This program can be extended, e.g. there can be discussed linear differential equations by using operators (LIERMANN 1966).

In certain analysis courses (in German High Schools) it is usual only to treat polynomial functions or rational functions. In this case it could be suitable to restrict oneself on algebraic methods without using limits. Suggestions may be found in (LEVI 1968).

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